Applied Mathematics E-Notes, 5(2005), 60-65 © ISSN 1607-2510
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NEW PROGRESS IN ENUMERATION OF MIXED MODELS*

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Received 9 September 2004

Abstract

A mixed model is said to be constrained if none of its fixed factors is nested within a random factor. Hess and Iyer have asked the question of enumerating non-isomorphic mixed models (with or without constraint), solving the problem for mixed models of size at most 5. In this article, using previous results about posets, we present a new algorithm, and obtain results for mixed models of size at most 10.

1 Introduction

The number of non-isomorphic fixed effects ANOVA models is known for $n \leq 16$ [2], since there is a one-to-one correspondence with posets. Hess and Iyer have worked on the problem of enumerating non-isomorphic mixed models (with or without constraint). A mixed model is said to be constrained if no fixed factor is nested with a random factor. They obtained results for mixed models of size at most 5 [8], proving there are 576 mixed models with constraint.

Mixed model is a special case of the general linear model, where both the mean function and covariance matrix for a data have a linear structure. More precisely, the vector of observations $\mathbf{y}$ is assumed to have moments:

$$E[\mathbf{y}] = \mathbf{X}\alpha$$

$$Var[\mathbf{y}] = \sum_t \phi_t \mathbf{V_t}$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)^t$ and $\phi = (\phi_1, \ldots, \phi_q)^t$ are unknown parameters and the matrices $\mathbf{X}, \mathbf{V_1}, \ldots, \mathbf{V_q}$ are given. Any mean vector and covariance matrix can be written in

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this form, but we are interested in models in which the parameters $\alpha_1, \ldots, \alpha_p, \phi_1, \ldots, \phi_q$ are functionally independent.

Hess and Iyer have written a SAS macro which allows to determine confidence intervals so as to facilitate the calculations of the Satterwaite method using the implementation of Burdick and Graybill [3]. Efficiency domain of their macro is reduced to models with at most 5 factors. We announced in [1] that we have succeeded in enumerating constrained and unconstrained mixed models up to 9 factors. We present here our work, which uses previous works about posets [7, 4, 5], and we extend the results, at the present time, to constrained mixed models with at most 10 factors.

2 Propositions

We start with the following correspondence which was established in [8]:

THEOREM 1. There exists a one-to-one correspondence between fixed effects models and posets $(P, <)$.

PROOF. Let us consider a poset P with n elements and a fixed effects model F with n factors. Crossed factors of the mixed model correspond to incomparable elements of the poset. A factor j is nested within a factor i in the mixed model if, and only if, we have $j < i$ in the poset P.

Thanks to this correspondence, the number of non-isomorphic fixed effects models of size at most n is known [2].

For mixed models, we introduce the notion of proset, which stands for Partially Reflexive Ordered SET.

DEFINITION 1. A proset is a set with a transitive and anti-symmetric binary relation.

REMARK 1. One can think a proset P_r as a poset P for which the order relation has been slightly modified, so that it can be reflexive for some elements of P only. In other words, we allow diagonal coefficients of the incidence matrix of the proset P_r to be one or zero. In this case, we shall say that the proset P_r induces the poset P.

THEOREM 2. There is a one-to-one correspondence between mixed models (without constraint) and prosets.

PROOF. Let us consider a proset P with n elements and a mixed model M with n elements. For distinct elements of the proset, it is the same situation as before. A reflexive element of P corresponds to a random factor of M, and a non-reflexive element of P corresponds to a fixed factor of M.

We have a natural notion of duality within prosets.

DEFINITION 2. A proset P' is the dual of the proset P if, and only if:

- P and P' induce the same poset P.·
- the reflexive elements of P' are exactly the non-reflexive elements of P.

With incidence matrix, P' is obtained from P by changing 0 into 1 and 1 into 0 on the diagonal of the incidence matrix of P.

REMARK 2. Let n be an odd integer. By duality, we can say that the number of mixed models of size n without constraint is even.

Hess and Iyer managed to enumerate non-isomorphic mixed models of size at most 5, with and without constraint. Our algorithm has allowed us to extend these results for $n \leq 10$.

3 Enumeration of mixed models

We represent a proset $M = (x_1, x_2, ..., x_n, \prec)$ by its incidence matrix $(m_{ij}) \in \{0,1\}^{n^2}$ with $m_{ij} = 1$ if, and only if, $x_i \prec x_j$. Not only is this a good representation in terms of implementation, but also it leads us to index and sort the vertices. Our algorithm aims at minimising the number of permutations of the vertices to detect isomorphic prosets.

Hess and Iyer algorithm:

1. Generation of adjacency matrices of posets of size n.

2. For each poset, construction of the set of matrix obtained by all possible combinations of 0 and 1 on the diagonal.

3. If only constrained mixed models are wanted, suppression of any matrices A which violates the constraint, by computing $A' = \sum_{i=0}^{i=n-1} A^k$ and by checking whether $A'(i,j) \geq 1$ for any couple (i,j) such as $A(i,i) = 1$ and $A(j,j) = 0$.

4. Suppression of isomorphic models.

Our algorithm:

1. Generation of adjacency matrices of posets of size n with the algorithm of enumeration of Chaunier and Lygeros [4, 5].

2. For each poset, construction of the incidence matrix A obtained by putting 0 and 1 on the diagonal such as the number of 1 is at most $\frac{n}{2}$. This way, we generate only one proset for each couple of dual prosets, thus reducing the number of isomorphism tests. Note that there are 2^{n-1} such matrices if n is odd and $2^{n-1} + \binom{n}{\frac{n}{2}}$ if n is even.

3. Elimination of all isomorphic prosets.

4. Incrementing by 2 the number of mixed models without constraint. Then we check whether the mixed model, and its dual, satisfy the constraint or not.

REMARK 3. So as to check whether a proset validates the constraint or not, it is sufficient to check whether a reflexive element is adjacent to a non-reflexive element. As our algorithm creates canonical prosets (via the generation algorithm of posets), this test is elementary.

REMARK 4. If only the number of constrained mixed models is wanted, one can eliminate mixed models that violate the constraint before doing the isomorphism test.

4 Results and interpretation.

First we remind known results about the enumeration of non-isomorphic posets, and then we give the results obtained for mixed models.

Table 1: Number of fixed effects models

Factors	Fixed effects models	Names and years
$n = 1$	1	
$n = 2$	2	
$n = 3$	5	
$n = 4$	16	
$n = 5$	63	
$n = 6$	318	
$n = 7$	2.045	Wright 1972
$n = 8$	16.999	Das 1977
$n = 9$	183.231	Mohring 1984
$n = 10$	2.567.284	Culberson, Rawlins 1990
$n = 11$	46.749.427	Culberson, Rawlins 1990
$n = 12$	1.104.891.746	C. Chaunier, N. Lygeros 1991
$n = 13$	33.823.827.452	C. Chaunier, N. Lygeros 1994
$n = 14$	1.338.193.159.771	N. Lygeros, P. Zimmermann 2000
$n = 15$	68.275.077.901.156	G. Brinkmann, B. D. McKay 2002
$n = 16$	44.831.306.651.195.087	G. Brinkmann, B. D. McKay 2002

Table 2: Number of mixed models (Not Constrained)

Factors	Mixed models (Not Constrained)	Names and years
$n = 1$	2	A. Hess, H. Iyer 1999
$n = 2$	7	A. Hess, H. Iyer 1999
$n = 3$	32	A. Hess, H. Iyer 1999
$n = 4$	192	A. Hess, H. Iyer 1999
$n = 5$	1.490	A. Hess, H. Iyer 1999
$n = 6$	15.067	R. Bayon, N. Lygeros, J.-S. Sereni 2002
$n = 7$	198.296	R. Bayon, N. Lygeros, J.-S. Sereni 2002
$n = 8$	3.398.105	R. Bayon, N. Lygeros, J.-S. Sereni 2002
$n = 9$	75.784.592	R. Bayon, N. Lygeros, J.-S. Sereni 2002

Table 3: Number of constrained mixed models

Factors	Mixed models (Constrained)	Names and years
$n = 1$	2	A. Hess, H. Iyer 1999
$n = 2$	6	A. Hess, H. Iyer 1999
$n = 3$	22	A. Hess, H. Iyer 1999
$n = 4$	101	A. Hess, H. Iyer 1999
$n = 5$	576	A. Hess, H. Iyer 1999
$n = 6$	4.162	R. Bayon, N. Lygeros, J.-S. Sereni 2002
$n = 7$	38.280	R. Bayon, N. Lygeros, J.-S. Sereni 2002
$n = 8$	451.411	R. Bayon, N. Lygeros, J.-S. Sereni 2002
$n = 9$	6.847.662	R. Bayon, N. Lygeros, J.-S. Sereni 2002
$n = 10$	133.841.440	R. Bayon, N. Lygeros, J.-S. Sereni 2003

Now, we underline the relation between the number of fixed effects ANOVA models with n elements and the order of the automorphism group of the associated poset [6, 9]. In the following, $M_{P_m} \times K$ means that there are M posets with automorphism groups of order m, each generating K mixed models (without constraint).

$M_1 = 2 = 1_{P_1} \times 2$
$M_2 = 7 = 1_{P_1} \times 4 + 1_{P_2} \times 3$
$M_3 = 32 = 2_{P_1} \times 8 + 2_{P_2} \times 6 + 1_{P_6} \times 4$
$M_4 = 192 = 5_{P_1} \times 16 + (6_{P_2} \times 12 + 1_{P_2} \times 10) + 1_{P_4} \times 9 + 2_{P_6} \times 8 + 1_{P_{24}} \times 5$

We see here that the order of the automorphism group of the poset is not sufficient to know the number of mixed models generated. The smallest posets to illustrate this have 4 elements:

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Both have an automorphism group of order 2, but P_0 generates 10 mixed models whereas P_1 generates 12. This is due to the structure of the automorphism group: $A(P_0) = \{id, (12)(34)\}$ whereas $A(P_1) = \{id, (12)\}$.

More precisely:

PROPOSITION 1. Let $(P, <)$ be a poset of size n with automorphism group G of order a.

(i) If $a = 1$ then P generates 2^n mixed models (without constraint).

(ii) If $a = n!$ then P generates $n + 1$ mixed models (without constraint).

(iii) If we know the composition of G, we can compute the number of mixed models (without constraint) generated by P, thanks to Pólya Theorem [10]. This number is:

$$\mathcal{Z}_G(2) = \frac{1}{a} \sum_{g \in G} 2^{\lambda_1(g)+\lambda_2(g)+\cdots+\lambda_n(g)}$$

where $\lambda_i(g)$ is the number of disjoint cycles of size i in the decomposition of g in disjoint cycles.

Last, the result of Prömel [11] on rigid posets, that is almost all posets have trivial automorphism group, explicitly shows us the efficiency of our method. By a structure isomorphism, it is easily seen that almost all prosets are rigid, and that proportionally, prosets have an effective density which is superior for a given n.

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