On the iso-hyper-representation theory

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Abstract

The hyper-representation theory appeared in mid 80's. Nowdays, it is clear that this theory is refered to the very large class of hyperstructures since the H_v-structures are used. The main problem is that only few theorems, from the classical representation theory, can be transferred to hyperstructures. However, the main results prove that there is a strong relation with the fundamental structures corresponding to each H_v-structure. Moreover, one can have results if the e-hyperstructures are used, that is the hyperstructures which are appropriate to Santilli's iso-theory. We present the general problem and we give some results and applications on the topic.

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1. Introduction

The main object of this paper is the class of hyperstructures called *H_v-structures* introduced in 1990 [10], which satisfy the *weak axioms* where the non-empty intersection replaces the equality. Some basic definitions:

Algebraic hyperstructure is called a set H equipped with one *hyperoperation* (abbreviation: *hyperoperation=hope*) $\cdot: H\times H\to P(H)-\{\varnothing\}$. Abbreviate by *WASS* the *weak associativity*: $(xy)z\cap x(yz)\neq\varnothing$, $\forall x,y,z\in H$ and by *COW* the *weak commutativity*: $xy\cap yx\neq\varnothing$, $\forall x,y\in H$. The hyperstructure $(H,\cdot)$ is called *H_v-semigroup* if it is *WASS,* it is called *H_v-group* if it is reproductive H_v-semigroup, i.e. $xH=Hx=H$, $\forall x\in H$.

Motivation. The quotient of a group with respect to an invariant subgroup, is a group. F. Marty 1934, states that, the quotient of a group by a subgroup is a hypergroup. Finally, the quotient of a group by a partition (or equivalently, by an equivalence relation) is an H_v-group [10], [11].

In an H_v-semigroup the *powers* are defined by: $h^1=\{h\}$, $h^2=h\cdot h,\dots$, $h^n=h^\circ h^\circ\dots{}^\circ h$, where $(^\circ)$ is the *n-ary circle hope,* i.e. take the union of hyperproducts, n times, with all possible patterns of parentheses put on them. An H_v-semigroup $(H,\cdot)$ is *cyclic of period s*, if there is an h, called *generator*, and a natural s, the minimum: $H=h^1\cup h^2\cup\dots\cup h^s$.

Analogously the cyclicity for the infinite period is defined [11]. If there is an h and s, the minimum: $H=h^s$, then (H,·) is called *single-power cyclic of period s*.

In an a similar way more complicated hyperstructures are defined:

(R,+,·) is called *H_v-ring* if (+) and (·) are *WASS*, the reproduction axiom is valid for (+) and (·) is *weak distributive* with respect to (+):

$$x(y+z)\cap(xy+xz)\neq\varnothing,\quad (x+y)z\cap(xz+yz)\neq\varnothing,\quad \forall x,y,z\in R.$$

Let (R,+,·) be an H_v-ring, (M,+) be a COW H_v-group and there is an external hope

$$\cdot:\ R\times M \rightarrow P(M): (a,x)\rightarrow ax$$

such that, $\forall a,b\in R$ and $\forall x,y\in M$, we have

$$a(x+y)\cap(ax+ay)\neq\varnothing,\ (a+b)x\cap(ax+bx)\neq\varnothing,\ (ab)x\cap a(bx)\neq\varnothing,$$

then M is called an *H_v-module* over F. In the case of an H_v-field F, which is defined later, instead of an H_v-ring R, then the *H_v-vector space* is defined.

For more definitions and applications on H_v-structures one can see [2],[4],[11].

Let (H,·), (H,*) be H_v-semigroups on the same set H, the hope (·) is called *smaller* than the (*), and (*) *greater* than (·), iff there exists an

$$f\in Aut(H,*)\quad \text{such that}\quad xy\subset f(x*y),\ \forall x,y\in H.$$

Then we write $\cdot\leq *$ and we say that (H,*) *contains* (H,·). If (H,·) is a structure then it is called *basic structure* and (H,*) is called *H_b-structure*.

Theorem 1.1 *(Little Theorem).* Greater hopes than the ones which are WASS or COW, are also WASS or COW, respectively.

This Theorem leads to a partial order on H_v-structures and to a correspondence between them and posets. Thus, we obtain an extreme large number of H_v-structures just enlarging the results.

Let (H,·) be hypergroupoid. We *remove* $h\in H$, if we take the restriction of (·) in the set H-{h}. $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h by $\underline{h}$ and h does not appear. $\underline{h}\in H$ *merges* with $h\in H$, if we take as product of any $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class with representative $\underline{h}$ [14].

An interesting class of H_v-structures is the following [11]:

Definition 2.1 An H_v-structure is called *very thin* iff all hopes are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in H there exists a hope (·) and a pair $(a,b)\in H^2$ for which ab=A, with cardA>1, and all the other products, are singletons.

To compare classes we can see the small sets. In the problem of enumeration and classification of H_v-structures we have interesting results by using computers. The partial order restricts the problem in finding the minimal, *up to isomorphisms*, H_v-

structures. We have results by Bayon & Lygeros as the following [1]: Let H={a,b} be a set of two elements, there are 20 H_v-groups, up to isomorphism. Up to isomorphism, in sets with three elements there are 6.494 minimal H_v-groups. 137 are abelians and 6.357 are not; 6.152 are cyclic and 342 are not. The number of H_v-groups with three elements is 1.026.462. The 7.926 are abelians, 1.018.536 are not; 1.013.598 are cyclic and 12.864 are not, 16 are very thin.

2. Fundamental relations

The main tool in hyperstructures is the *fundamental relation.* M. Koscas, in 1970, defined in hypergroups the relation β and its transitive closure β^*. This relation connects hyperstructures with the classical structures and is defined in H_v-groups as well. T. Vougiouklis [9], [10], [11], [12] introduced the γ^* and ε^* relations, which are defined, in H_v-rings and H_v-vector spaces, respectively. He also named these relations, *fundamental*.

Definition 2.1 The fundamental relations β^*, γ^* and ε^*, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively.

Specifying the above motivation we remark: Let $(G,\cdot)$ be a group and R be an equivalence relation (or a partition) in G, then $(G/R,\cdot)$ is an H_v-group, therefore the quotient $(G/R,\cdot)/\beta^*$ is a group, the *fundamental* one. The classes of $(G/R,\cdot)/\beta^*$ are a union of some of the R-classes.

The way to find the fundamental classes is given by the following:

Theorem 2.2 Let $(H,\cdot)$ be an H_v-group and denote by U the set of all finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then β^* is the transitive closure of β.

Analogous theorems are for H_v-rings, H_v-vector spaces and so on [11], [18].

An element is called *single* if its fundamental class is singleton [11].

More general structures can be defined by using the fundamental structures. An application in this direction is the general hyperfield. There was no general definition of a hyperfield, but from 1990 [10] there is the following [11]:

Definition 2.3 An H_v-ring $(R,+,\cdot)$ is called *H_v-field* if R/γ^* is a field.

Since the algebras are defined on vector spaces, the analogous to Theorem 2.2, on H_v-vector spaces is the following:

Theorem 2.4 Let $(V,+)$ be an H_v-vector space over the H_v-field F. Denote by U the set of all expressions consisting of finite hopes either on F and V or the external hope applied on finite sets of elements of F and V. We define the relation ε, in V as follows: $x\varepsilon y$ iff $\{x,y\}\subset u$ where $u\in U$. Then the relation ε^* is the transitive closure of the relation ε.

Definition 2.5 [11], [18]. Let $(L,+)$ be H_v-vector space over the H_v-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Let ω_L be the core of the canonical map $\varphi':L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Consider the *bracket (commutator) hope*:

$$[\ ,\ ]: L\times L\to P(L): (x,y)\to [x,y]$$

then L is an *H_v-Lie algebra* over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y]) \neq \emptyset$

$[x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2]) \neq \emptyset,\ \forall x,x_1,x_2,y,y_1,y_2\in L$ and $\lambda_1,\lambda_2\in F$

(L2) $[x,x]\cap\omega_L \neq \emptyset,\ \forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L \neq \emptyset,\ \forall x,y\in L$

This is a general definition so one can use special cases in order to face problems in applied sciences.

The definition of enlarged hyperstructures, introduces a new class [14]:

Definition 2.6 The H_v-semigroup $(H,\cdot)$ is called *h/v-group* if H/β^* is a group.

The class of h/v-groups is more general than the H_v-groups since in h/v-groups the reproductivity is not valid. However, the *reproductivity of classes* is valid, i.e. if H is partitioned into equivalence classes $\sigma(x)$, then $x\sigma(y)=\sigma(xy)=\sigma(x)y,\ \forall x,y\in H$ [24]. This is so because the quotient is reproductive. In a similar way the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc are defined.

Construction 2.7 [14]. Let $(H,\cdot)$ be an H_v-semigroup and $v\notin H$ and $(\underline{H},\cdot)$ be its attached h/v-group. Consider a $0\notin\underline{H}$ and define in $\underline{H}_o=H\cup\{v,0\}$ two hopes:

hyperaddition (+) and *hypermultiplication* (·), by the following multiplicative tables:

+	0	y	v
0	0	H	v
x	H	v	0
v	v	0	H

·	0	y	v
0	0	0	0
x	0	xy	v
v	0	v	H

Then $(\underline{H}_o,+,\cdot)$ is an h/v-field with $(\underline{H}_o,+,\cdot)/\gamma^*\cong\boldsymbol{Z}_3$. The hope (+) is associative, (·) is WASS and weak distributive with respect to (+). 0 is zero absorbing and single element but not scalar in (+). The $(\underline{H}_o,+,\cdot)$ is called the *Attached h/v-field* of the $(H,\cdot)$.

Denote by U the set of all finite products of elements of a hypergroupoid $(H,\cdot)$. Consider the relation defined as follows:

$$xLy \text{ iff there exists } u\in U \text{ such that } ux\cap uy\neq\varnothing.$$

Then the transitive closure L^* of L is called *left fundamental reproductivity relation.* Similarly, the *right fundamental reproductivity relation* R^* is defined.

Theorem 2.8 If (H,·) is a commutative semihypergroup, i.e. the strong commutativity and the strong associativity is valid, then the strong expression of the above L relation: $ux=uy$, has the property: $L^*=L$.

Theorem 2.9 Let $(H,\otimes)$ be an H_b-semigroup with commutative the basic semigroup (H,·), has at least one element $w\in H$ such that the set $w\cdot H$ is finite. Then $(H/L^*,\otimes)$, where L is the relation: xLy iff there exists $z\in H$ such that $zx=zy$ and $(\otimes)$ is the induced hope on classes, (i.e. L is defined with respect to (·)), is a finite commutative h/v-group.

The uniting elements method was introduced by Corsini–Vougiouklis [3] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties. The *uniting elements* method is the following: Let ***G*** be algebraic structure and d, a property which is not valid. Suppose that d is described by a set of equations; then, take the partition in ***G*** for which it is put together, in the same class, every pair of elements that causes the non-validity of the property d. The quotient by this partition ***G***/d is an H_v-structure. Then, quotient out the H_v-structure ***G***/d by the fundamental relation β^*, a stricter structure $(\boldsymbol{G}/d)\beta^*$ for which the property d is valid, is obtained.

It is very importand if more properties are desired. The reason is that, some of the properties lead straighter to the classes than others, thus, it is better to apply first them. One can do this because analogous to the following theorem is valid, for the several hyperstructures:

Theorem 2.10 [11]. Let $(\boldsymbol{R},+,\cdot)$ be a ring, and $F=\{f_1,\ldots, f_m, f_{m+1},\ldots, f_{m+n}\}$be a system of equations on ***R*** consisting of two subsystems $F_m=\{f_1,\ldots,f_m\}$ and $F_n=\{f_{m+1},\ldots, f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the ring $\boldsymbol{R}_m= (\boldsymbol{R}/\sigma_m)/\gamma^*$. Then

$$(\boldsymbol{R}/\sigma)/\gamma^* \cong (\boldsymbol{R}_m/\sigma_n)/\gamma^*.$$

3. Large classes of hyperstructures

A large class of H_v-structures is the following [17]:

Definition 3.1 Let (G,·) be groupoid (resp., hypergroupoid) and $f:G\to G$ be a map. We define a hope (∂), called *theta-hope,* we write *∂-hope*, on G as follows

$$x\partial y = \{f(x)\cdot y, x\cdot f(y)\},\ \forall x,y\in G,\ \ (\text{resp. } x\partial y= (f(x)\cdot y)\cup(x\cdot f(y)),\ \forall x,y\in G)$$

If (·) is commutative then ∂ is commutative. If (·) is *COW*, then ∂ is *COW*.

If (G,·) is a groupoid (or hypergroupoid) and $f:G\to P(G)-\{\varnothing\}$ be any multivalued map. We define the ∂-hope on G as follows: $x\partial y = (f(x)\cdot y)\cup(x\cdot f(y)),\ \forall x,y\in G$.

Let $(G,\cdot)$ be a groupoid, f_i: $G\rightarrow G$, $i\in I$, be a set of maps on G. The

$$f_{\cup}:G\rightarrow P(G): f_{\cup}(x)=\{f_i(x) \mid i\in I\},$$

is *the union* of $f_i(x)$. We have the *union ∂-hope* (∂), on G if we take $f_{\cup}(x)$. If $\underline{f} \equiv f\cup(id)$, then we have the *b-∂-hope*.

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. The basic property is that if $(G,\cdot)$ is a semigroup then for every f, the ∂-hope is *WASS*.

Several results can be obtained using ∂-hopes [17]:

Example. Consider the group of integers $(\boldsymbol{Z},+)$ and $n\neq 0$ be a natural number. Take the map f such that $f(0)=n$ and $f(x)=x$, $\forall x\in \boldsymbol{Z}-\{0\}$. Then $(\boldsymbol{Z},\partial)/\beta^* \cong (\boldsymbol{Z}_n,+)$.

Theorems 3.2 (a) In integers $(\boldsymbol{Z},+,\cdot)$ fix $n\neq 0$, a natural number. Consider the map f such that $f(0)=n$ and $f(x)=x$, $\forall x\in \boldsymbol{Z}-\{0\}$. Then $(Z, \partial_+, \partial\cdot)$, where ∂_+ and $\partial\cdot$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with

$$(\boldsymbol{Z},\partial_+,\partial\cdot)/\gamma^* \cong \boldsymbol{Z}_n.$$

(b) In $(\boldsymbol{Z},+,\cdot)$ with $n\neq 0$, take f such that $f(n)=0$ and $f(x)=x$, $\forall x\in \boldsymbol{Z}-\{n\}$. Then $(\boldsymbol{Z},\partial_+,\partial\cdot)$ is an H_v-ring, moreover, $(\boldsymbol{Z},\partial_+,\partial\cdot)/\gamma^* \cong \boldsymbol{Z}_n$.

Special case of the above is for n=p, prime, then $(\boldsymbol{Z},\partial_+,\partial\cdot)$ is an H_v-field.

Theorem 3.3 Let $(\boldsymbol{V},+,\cdot)$ be an algebra over the field $(F,+,\cdot)$ and $f:\boldsymbol{V}\rightarrow \boldsymbol{V}$ be a map. Consider the ∂-hope defined only on the multiplication of the vectors (·), then $(\boldsymbol{V},+,\partial)$ is an H_v-algebra over F, where the related properties are weak. If, moreover f is linear then we have more strong properties.

Theorem 3.4 Let $(\boldsymbol{A},+,\cdot)$ be an algebra over the field F. Take any map f: $\boldsymbol{A}\rightarrow\boldsymbol{A}$, then the *∂-hope on the Lie bracket* $[x,y]=xy-yx$, is defined as follows

$$x\partial y = \{f(x)y-f(y)x,\ f(x)y-yf(x),\ xf(y)-f(y)x,\ xf(y)-yf(x)\}.$$

then $(\boldsymbol{A},+,\partial)$ is an H_v-algebra over F, with respect to the ∂-hopes on Lie bracket, where the weak anti-commutativity and the inclusion linearity is valid.

An important large class of hyperstructures created from classical structured is the following [9],[11],[12],[13]:

Definition 3.5 Let $(G,\cdot)$ be groupoid, then for every $P\subset G$, $P\neq\emptyset$, we define the following hopes called *P-hopes*:

$$\underline{P}: x\underline{P}y=(xP)y\cup x(Py),\quad \underline{P}_r: x\underline{P}_ry=(xy)P\cup x(yP),\quad \underline{P}_l: x\underline{P}_ly=(Px)y\cup P(xy),\ \forall x,y\in G.$$

The $(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called *P-hyperstructures*. The most usual case is if $(G,\cdot)$ is semigroup, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and $(G,\underline{P})$ is a semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS.

A generalization of P-hopes, needed in Santilli's isotheory, is the following [5]:

Construction 3.6 Let (G,·) be an abelian group and P any subset of G with more than one elements. We define the hope ($\times_P$) as follows:

$$x\times_P y = \begin{cases} x\cdot P\cdot y = \{x\cdot h\cdot y \mid h\in P\} & \text{if } x\neq e \text{ and } c\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

we call this hope P_e-hope. The hyperstructure (G,$\times_P$) is abelian H_v-group.

4. Representations of hyperstructures

In the classical theory of representations we have the following basic definitions: Let **G** be a group and $\mathcal{V}$ be a finite dimensional vector space over the field **F**. A *representation* of **G** is a homomorphism ρ: **G**→Aut($\mathcal{V}$) of **G** into the set of automorphisms of $\mathcal{V}$. Analogous definitions are given for complicate structures: Let $\mathcal{L}$ be a Lie algebra then a *rep* of $\mathcal{L}$ is a homomorphism ρ: $\mathcal{L}\to\mathcal{gl}(\mathcal{V})$, from $\mathcal{L}$ into linear transformations on $\mathcal{V}$ over **F**. Since there exists 1-1 correspondence on the sets of all endomorphisms with n×n matrices, where n=dim$\mathcal{V}$, any representation corresponds to each element, of a finite group, a matrix, and this set of matrices acts exactly as the group. With this theory, mathematicians try to transfer the study of structures into the study of matrices which is clear and easy. Ado's theorem states that all finite dimensional Lie algebras have a faithful finite dimensional representation. The two steps in representation theory: first, by the Cayley's theorem every group has a faithful representation by permutations. Second, every permutation group of order n can be represented by n×n *monomial matrices*. The above steps are clear but the obtained representations are not useful since the matrices are of type n×n. Thus, the main attempt is to reduce the dimension of representations. Most important is to find the irreducible representations over the field of real or complex numbers. The representation theory represents all groups in one form so that they can be compared and studied in the same way. Thus the low dimensional representations are most useful.

The corresponding theory on hyperstructures is the representation theory of H_v-groups by H_v-matrices.

Definitions 4.1 [11], [13], [15], [16]. *H_v-matrix* is called a matrix with entries elements of an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices (a_{ij}) and (b_{ij}), of type m×n and n×r respectively, is defined, in the usual manner but it is a set of m×r H_v-matrices. The sum of products of elements of the H_v-ring is the union of the sets obtained with all possible parentheses put on them, called *n-ary circle hope* on the hyperaddition. The hyperproduct of H_v-matrices is not nessesarily *WASS*.

The problem of the H_v-matrix representations is the following:

Let $(H,\cdot)$ be H_v-group. Find an H_v-ring $(R,+,\cdot)$, a set $M_R=\{(a_{ij}) \mid a_{ij}\in R\}$ and a map

$$T: H\to M_R: h\mapsto T(h) \text{ such that } T(h_1h_2)\cap T(h_1)T(h_2)\neq\varnothing, \forall h_1,h_2\in H.$$

The map T is called *H_v-matrix* representation.

If the $T(h_1h_2)\subset T(h_1)(h_2)$, $\forall h_1,h_2\in H$ is valid, then T is called *inclusion* representation.

If $T(h_1h_2)=T(h_1)(h_2)=\{T(h) \mid h\in h_1h_2\}$, $\forall h_1,h_2\in H$, then T is called *good representation*.

If T is one to one and good then it is a *faithful* representation.

The problem of representations is complicated because the cardinality of the product of H_v-matrices is very big. But it can be simplified in special cases such as the following:

(a) The H_v-matrices are over H_v-rings with 0 and 1 and if these are scalars.

(b) The H_v-matrices are over *very thin* H_v-rings.

(c) The case of 2×2 H_v-matrices.

(d) The case of H_v-rings which contains singles, then these act as absorbings.

The main theorem of representations is the following [19]:

Theorem 4.2 A necessary condition in order to have an inclusion representation T of an H_v-group $(H,\cdot)$ by n×n H_v-matrices over the H_v-ring $(R,+,\cdot)$ is the following:

For all classes $\beta^*(x)$, $x\in H$ there must exist elements $a_{ij}\in H$, $i,j\in\{1,...,n\}$ such that

$$T(\beta^*(a)) \subset \{A= (a'_{ij}) \mid a'_{ij}\in\gamma^*(a_{ij}),\ i,j\in\{1,...,n\}\}$$

Theorem 4.3 Every inclusion representation $T: H\to M_R: a\mapsto T(a)=(a_{ij})$ of an H_v-group $(H,\cdot)$ by n×n H_v-matrices over the H_v-ring $(R,+,\cdot)$, induces an homomorphic n×n representation T* of the fundamental group H/β^* over the fundamental ring R/γ^* by setting

$$T^*(\beta^*(a))=[\gamma^*(a_{ij})], \quad \forall\beta^*(a)\in H/\beta^*,$$

where the element $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. Then T* is called *fundamental induced representation* of T.

Denote $tr_\varphi(T(x)) = \gamma^*(T(x_{ii}))$ the fundamental trace, then the mapping

$$X_T: H\to R/\gamma^*: x\mapsto X_T(x)= tr_\varphi(T(x))= trT^*(x)$$

is called *fundamental character*.

For an attached H_v-field ($\underline{H}_o$, +, ·), in $\Sigma a_{ik}\cdot b_{kj}$ the terms $a_{ik}\cdot b_{kj}$ could be 0,v,x or H (where x∈H). But any sum is only 0 or v or H. Thus, for finite H_v-fields ($\underline{H}_o$, +, ·), if the set H appears in t entries then the cardinality of the hyperproducts is $(cardH)^t$.

The main attached h/v-fields give hyperfields where the cardinality of products is small, since 0 is absorbing. Removing, absorbing and merging we reduce the cardinality.

Constructions 4.4

(i) Let (H,·) be a H_v-group, then for every (⊕) such that $x\oplus y\supset\{x,y\}$, ∀x,y∈H, the (H,⊕,·) is an H_v-ring. These H_v-rings are called *associated to* (H,·) H_v-rings. In representation theory of hypergroups, in sense of Marty, there are three associated hyperrings (H,⊕,·) to (H,·). The (⊕) is defined respectively, ∀x,y∈H, as follows:

type a: $x\oplus y=\{x,y\}$, type b: $x\oplus y=\beta^*(x)\cup \beta^*(y)$, type c: $x\oplus y=H$.

In the above types the strong associativity and strong distributivity, is valid.

(ii) Let (H,+) be H_v-group. Then for every hope (⊗) such that $x\otimes y\supset\{x,y\}$, ∀x,y∈H, the hyperstructure (H,+,⊗) is an H_v-ring.

We conclude with some open problems on representations on hypergroups:

(a) Find standard H_v-rings and H_v-fields to represent all H_v-groups by H_v-matrices.

(b) Find representations by H_v-matrices over standard finite H_v-rings analogous to Z_n.

(c) Find the 'minimal' hypermatrices corresponding to the minimal hopes.

5. Applications on Santilli's iso-theory

Last decades H_v-structures have applications in other branches of mathematics and in sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, hyperstructures can be widely applicable in industry and production, too [2], [4], [5], [7], [8].

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit [6]. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli - Vougiouklis in 1996 [7] and they are called *e-hyperfields*. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology.

Definition 5.1 A hyperstructure (H,·) which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

Definition 5.2 A hyperstructure (F,+,·), where (+) is an operation and (·) is a hope, is called *e-hyperfield* if the following axioms are valid: (F,+) is an abelian group with the additive unit 0, (·) is WASS, (·) is weak distributive with respect to (+), 0 is absorbing element: $0\cdot x=x\cdot 0=0$, $\forall x\in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x$, $\forall x\in F$, and $\forall x\in F$ there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield,* [5], [8], [18], [19].

Definition 5.3 *Main e-Construction.* Given a group (G,·), where e is the unit, then we define in G, a large number of hopes (⊗) as follows:

$$x\otimes y = \{xy, g_1, g_2,\ldots\}, \forall x,y\in G-\{e\}, \text{ and } g_1, g_2,\ldots\in G-\{e\}$$

$g_1, g_2,\ldots$ are not necessarily the same for each pair (x,y). (G,⊗) is an H_v-group, it is an H_b-group which contains the (G,·). (G,⊗) is an e-hypergroup. Moreover, if for each x,y such that xy=e, so we have $x\otimes y=xy$, then (G,⊗) becomes a strong e-hypergroup.

The proof is immediate ftom the Little Theorem. Moreover one can see that the unit e is a unique scalar and for each x in G, there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$. If the $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then (G,⊗) is strong e-hypergroup.

The main e-construction gives an extremely large number of e-hopes.

Example 5.4 Consider the quaternion group

Q={1,-1, i,-i, j,-j, k,-k} with defining relations $i^2 = j^2 = -1$, $ij = -ji = k$.

Denoting $\underline{i}$={i,-i}, $\underline{j}$={j,-j}, $\underline{k}$={k,-k} we may define a very large number (∗) hopes by enlarging only few products. For example, $(-1)*k=\underline{k}$, $k*i=\underline{j}$ and $i*j=\underline{k}$. Then the hyperstructure (Q,∗) is a strong e-hypergroup.

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