The LV-hyperstructures in Santilli's iso-theory

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Abstract

The Level Variable (LV) construction is a hyperstructure containing a hyperoperation defined on every graded finite poset. More precisely, on graded posets with s levels we can use H_v-groups, one of them is an e-H_v-group S, needed in Santilli's iso-theory, where we define an LV-hyperoperation and we obtain an e-H_v-group. The important result is that the fundamental structure is isomorphic to the fundamental group of the initial e-H_v-group S. We extend these constructions to H_v-fields in order to have 'hypernumbers' appropriate in Santilli's iso-theory. Finally, we face the problem of enumeration of such constructions defined on finite small sets.

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1. Fundamental Definitions

In a set H is called *hyperoperation* (abbreviation *hyperoperation=hope*) in a set H, is called any map $\cdot : H\times H \rightarrow P(H)-\{\varnothing\}$.

Definition 1.1 *Marty 1934.* A hyperstructure (H,·) is a *hypergroup* if (·) is an associative hyperoperation for which the reproduction axiom: $hH=Hh=H$, $\forall x\in H$, is valid.

Definition 1.2 *Vougiouklis 1990.* In a set H with a hope we abbreviate by *WASS* the *weak associativity*: $(xy)z\cap x(yz)\neq\varnothing$, $\forall x,y,z\in H$ and by *COW* the *weak commutativity*: $xy\cap yx\neq\varnothing$, $\forall x,y\in H$. The hyperstructure (H,·) is called *H_v-semigroup* if it is *WASS,* it is called *H_v-group* if it is reproductive H_v-semigroup, i.e. $xH=Hx=H$, $\forall x\in H$. The hyperstructure (R, +,·) is called *H_v-ring* if both (+) and (·) are *WASS*, the reproduction axiom is valid for (+) and (·) is *weak distributive* with respect to (+): $x(y+z)\cap(xy+xz)\neq\varnothing$, $(x+y)z\cap(xz+yz)\neq\varnothing$, $\forall x,y,z\in R$.

Definition 1.3 *Santilly-Vougiouklis*. A hyperstructure (H,·) which contain a unique scalar unit e, is called e-hyperstructure. A hyperstructure (F, +,·), where (+) is an operation and (·) is a hyperoperation, is called *e-hyperfield* if the following axioms are valid:

1. (F,+) is an abelian group with the additive unit 0,
2. (·) is WASS,
3. (·) is weak distributive with respect to (+),
4. 0 is absorbing element: $0\cdot x=x\cdot 0=0$, $\forall x\in F$,
5. There exists a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1 = x$, $\forall x\in F$,
6. For every $x\in F$ there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Construction1.4 *Main e-Construction.* Given a group (G,·), where e is the unit, then we define in G, a large number of hyperoperations (⊗) as follows:

$$x\otimes y = \{xy, g_1, g_2,\ldots\}, \forall x,y\in G-\{e\}, \text{ and } g_1, g_2,\ldots\in G-\{e\}$$

$g_1, g_2,\ldots$ are not necessarily the same for each pair (x,y). Then (G,⊗) becomes an H_v-group, in fact is H_b-group which contains the (G,·). The H_v-group (G,⊗) is an e-hypergroup. Moreover, if for each x, y such that xy=e, so we have $x\otimes y=xy$, then (G,⊗) becomes a strong e-hypergroup.

The main tool to study hyperstructures are the *fundamental relations* β*, γ* and ε*, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, resp., as the smallest equivalences so that the quotient would be group, ring and vector space, resp. Fundamental relations are used for general definitions. Thus, an H_v-ring (R, +,·) is called *H_v-field* if R/γ* is a field.

Definition 1.5 Let (H,·), (H,∗) be H_v-semigroups defined on the same set H. Then (·) is called *smaller* than (∗), and (∗) *greater* than (·), iff there exists an $f\in Aut(H,*)$ such that $xy\subset f(x*y)$, $\forall x,y\in H$. Then we write $\cdot\leq *$ and we say that (H,∗) *contains* (H,·). If (H,·) is a structure then it is called *basic structure* and (H,∗) is called *H_b-structure*.

Theorem 1.6 *(The Little Theorem).* Greater hopes than the ones which are *WASS* or *COW*, are also *WASS* or *COW*, respectively.

This Theorem leads to a partial order on H_v-structures, thus we have posets. The determination of all H_v-groups and H_v-rings is very interesting. To compare classes we can see the small sets. The problem of enumeration of classes of H_v-structures was started very early but recently we have results by using computers. The partial order in H_v-structures restricts the problem in finding the minimals.

2. Enumeration Theorems

Theorem 2.1 *Chung-Choi.* There exists up to isomorphism, 13 minimal H_v-groups of order 3 with scalar unit, i.e. minimal e-hyperstructures of order 3.

Theorems 2.2 *Bayon-Lygeros.*

► There exists up to isomorphism, 20 H_v-groups of order 2.

► There exists up to isomorphism, 292 H_v-groups of order 3 with scalar unit, i.e. e-hyperstructures of order 3.

► There exists up to isomorphism, 6494 minimal H_v-groups of order 3.

► There exists up to isomorphism, 1026462 H_v-groups of order 3.

Theorems 2.3 *Bayon-Lygeros.*

► There exists up to isomorphism, 631 609 H_v-groups of order 4 with scalar unit, i.e. e-hyperstructures of order 4.

► There exists up to isomorphism, 8.028.299.905 abelian Hv-groups of order 4.

Theorems 2.4 *Bayon-Lygeros.*

► The numbers of abelian H_v-groups of order 4 with scalar unit (i.e. abelian e-hyperstructures) in respect with their automorphism group are the following:

|Aut(Hv)|	1	2	3	4	6	8	12	24
	_	_	_	32	_	46	5510	626021

► They are 63 isomorphism classes of hyperrings of order 2.

► They are 875 isomorphism classes of H_v-rings of order 2.

► They are 33277642 isomorphism classes of hyperrings of order 3.

In all the above results we construct the poset of hyperstructures of order 2 and 3 in the sense of inclusion for hyperproducts. We compute the Betti numbers of the poset of Hv-groups of order 2 and we have the following results: (1, 5), (2, 4), (3, 6), (4, 4), (5, 1). We also compute the Betti numbers of the poset of hypergroups of order 3 and we have the following results : (1, 59), (2, 168), (3, 294), (4, 438), (5, 568), (6, 585), (7, 536), (8, 480), (9,358), (10, 245), (11,160), (12, 66), (13, 29), (14, 10), (15, 2), (16,1).

We explicitly compute the Cayley subtables of the minimal e-hyperstructures with H={e,a,b} and we have for the products (aa, ab, ba, bb) the following results: (b; e; e; a), (eb; a; a; e), (e; ab; ab; e), (a; eb; eb; a), (ab; ea; ea; e), (H; eb; a; ea), (H; a; eb; ea), (a; H; H; e), (b; H; H; e), (a; H; H; b), (H; b; a; H), (H; a; b; H), (H; e; ab; H).

3. Construction Theorems

There are several ways to organize such posets using hyperstructure theory. We present now a new construction on posets and we name this LV-construction since it is based on gradations where the Levels are used as Variable. Thus LV means Level Variable.

Theorem 3.1 *The LV-Construction I*

Consider the set $\mathbf{P_n}$ of all H_v-groups defined on a set of n elements. Take the following gradation on $\mathbf{P_n}$ based on posets:

Level 0 (or *grade* 0), denoted by $\boldsymbol{g_0}$, is the set of all minimals of $\mathbf{P_n}$. Level (grade) 1, denoted by $\boldsymbol{g_1}$, is the set of all H_v-groups obtained from minimals by adding one only element to anyone of the results of the products of two elements on the minimals of $\mathbf{P_n}$, i.e. of $\boldsymbol{g_0}$. Level 2 (or grade 2), denoted by $\boldsymbol{g_2}$, is the set of all H_v-groups obtained from minimals by adding only two elements to anyone of the results of the products of two elements of the minimals $\boldsymbol{g_0}$. Then inductively the Level k is defined, denoted by $\boldsymbol{g_k}$. In the case that an H_v-group is obtained by adding k_1 elements of one minimal and by adding k_2 elements of another minimal then we consider that it belongs to the Level $\min(k_1,k_2)$.

Denote by r the cardinality of the minimals, $|\boldsymbol{g_0}|=r$, and by s the number of levels. Take any H_v-group with r elements corresponding to the r elements of $\boldsymbol{g_0}$, so we have an H_v-group $(\boldsymbol{g_0},*)$. Then we define a hope on

$$\mathbf{P_n} = \boldsymbol{g_0} \cup \boldsymbol{g_1} \cup,\dots,\cup \boldsymbol{g_{s-1}},$$

as follows

$$x \otimes y = \begin{cases} x*y, \ \forall x,y \in \boldsymbol{g_0} \\ \boldsymbol{g_{\kappa+\lambda}}, \ \forall x \in \boldsymbol{g_\kappa},\ y \in \boldsymbol{g_\lambda}, \ \text{where } (\kappa,\lambda) \neq (0,0) \end{cases}$$

Then the hyperstructure $(\mathbf{P_n},\otimes)$ is an H_v-group where its fundamental group is isomorphic to $\boldsymbol{Z_s}$, thus we have

$$\mathbf{P_n}/\beta^* \approx \boldsymbol{Z}_s.$$

Proof. Let us correspond, numbered, the levels with the elements of $\boldsymbol{Z_s}$: $\boldsymbol{g_i} \to \underline{i}$, i=0,…, s-1.

From the definition of $(\otimes)$ any hyperproduct of elements from several levels, apart of $\boldsymbol{g_0}$, equals to only one special set of H_v-groups that constitute one level. Moreover we have

$$x \otimes y = \boldsymbol{g_0}, \ \forall x \in \boldsymbol{g_\kappa},\ y \in \boldsymbol{g_{-\kappa}}, \ \text{for any } \kappa \neq 0.$$

That means that the elements of $\boldsymbol{g_0}$ are β^*-equivalent. Therefore all elements of each level are β^*-equivalent and there are no β^*-equivalent elements from different levels. That proves that

$$\mathbf{P_n}/\beta^* \approx \boldsymbol{Z}_s. \quad \blacksquare$$

The above is a construction similar to the one from our book: *Hyperstructures and their Representations* [p.27].

A generalization of the above construction is the following:

Theorem 3.2 *The LV-Construction II*

Consider a graded finite poset with n elements: $\mathbf{P_n}=\boldsymbol{g_0}\cup\boldsymbol{g_1}\cup,\dots,\cup\boldsymbol{g_{s-1}}$, with s levels (grades) $\boldsymbol{g_0}, \boldsymbol{g_1},\dots, \boldsymbol{g_{s-1}}$, such that $\sum_{i=0}^{s-1}|\boldsymbol{g_i}|=n$. Denoting $|\boldsymbol{g_0}|=r$, we consider two H_v-groups $(\boldsymbol{E},\cdot)$ and $(\boldsymbol{S},*)$ such that $|\boldsymbol{E}|=r$, $|\boldsymbol{S}|=s$ and moreover $\boldsymbol{S}$ has a unit scalar and single element e. Then we take 1:1 maps from ***E*** onto $\boldsymbol{g_0}$ and from ***S*** onto $\{\boldsymbol{g_0}, \boldsymbol{g_1},\ldots, \boldsymbol{g_{s-1}}\}$, so we obtain two H_v-groups: $(\boldsymbol{g_0},\cdot)$ and $(\boldsymbol{G}=\{\boldsymbol{g_0}, \boldsymbol{g_1},\ldots, \boldsymbol{g_{s-1}}\},*)$ where $\boldsymbol{E}=\boldsymbol{g_0}$ corresponds to the scalar single element e. We define a hope on $\mathbf{P_n}$ as follows:

$$x\otimes y = \begin{cases} x\cdot y, & \forall x, y\in \boldsymbol{g_0} \\ \boldsymbol{g_\kappa} * \boldsymbol{g_\lambda}, & \forall \boldsymbol{g_\kappa}, \boldsymbol{g_\lambda} \in \boldsymbol{G}, \text{ where } (\kappa,\lambda)\neq(0,0) \end{cases}$$

Then the hyperstructure $(\mathbf{P_n},\otimes)$ is an H_v-group where its fundamental group is isomorphic to the fundamental group of $(\boldsymbol{S},*)$, therefore we have

$$(\mathbf{P_n},\otimes)/\beta^* \approx (\boldsymbol{S},*)/\beta^*.$$

Proof. From the reproductivity of $(\boldsymbol{G},*)$, for each $\boldsymbol{g_\kappa}$, $\kappa\neq 0$, there exists a $\boldsymbol{g_\lambda}$ such that $\boldsymbol{g_0}\in\boldsymbol{g_\kappa}*\boldsymbol{g_\lambda}$. But $\boldsymbol{g_0}$ is a single element of $(\boldsymbol{S},*)$, therefore we have $\boldsymbol{g_0}=\boldsymbol{g_\kappa}*\boldsymbol{g_\lambda}$. Then, by the definition, for any $x\in\boldsymbol{g_\kappa}$, $y\in\boldsymbol{g_\lambda}$ we have, $x\otimes y=\boldsymbol{g_0}$. Therefore, all the elements of $\boldsymbol{g_0}$ are β^*-equivalent. On the other side, from the definition, all elements of each level are β^*-equivalent and they are β^*-equivalent elements with different levels if and only if they are β^*-equivalent in $(\boldsymbol{G},*)$. In other wards they follow exactly the β^*-equivalence of $(\boldsymbol{G},*)$.
That proves that

$$(\mathbf{P_n},\otimes)/\beta^* \approx (\boldsymbol{S},*)/\beta^*. \quad \blacksquare$$

With this LV-construction we can define the poset for H_v-groups of order 2. So we get a non-connected poset with Betti numbers for the two subposets (1,4), (2,4), (3,1) and (1,1), (2, 4), (3,6).

The following LV-construction III is originated from the principal realization of the Infinite Dimensional Kac-Moody Lie Algebras of the type A_n and D_n given in Kac's book and our paper: On affine Kac-Moody Lie algebras. These algebras are graded where each level has some constant basis elements, which are obtained by sifting the basis elements of level zero, plus some extra basis elements. Therefore all levels have a number of corresponding 1:1 basis elements and some levels have some more elements.

Theorem 3.3 *The LV-Construction III*

Consider a graded finite poset with n elements: $\mathbf{P_n}=\boldsymbol{g_0}\cup\boldsymbol{g_1}\cup,\ldots,\cup\boldsymbol{g_{s-1}}$, with s levels (grades) $\boldsymbol{g_0}, \boldsymbol{g_1},\ldots, \boldsymbol{g_{s-1}}$, such that $\sum_{i=0}^{s-1}|\boldsymbol{g_i}|=n$ and we correspond the levels to the group $(Z_s,+)$. Denote $r = \min\{|\boldsymbol{g_0}|,|\boldsymbol{g_1}|,\ldots,|\boldsymbol{g_{s-1}}|\}$, then we select r elements from each level and we correspond 1:1 to the elements of the group $(Z_r,+)$, which we denote by $\underline{r}$. Moreover, we correspond the extra elements of each level, to the element $\underline{1}$ which may became a set. Therefore, we have a partition of the set $\mathbf{P_n}$, into classes $\boldsymbol{g_{\underline{\kappa},\underline{\lambda}}}$, where $\underline{\kappa}$ denote the level and $\underline{\lambda}$ denotes the corresponding element of $(Z_s,+)$. All classes are singletons, except possible, of the case where $\underline{\lambda}=\underline{1}$. We define a hope on $\mathbf{P_n}$ as follows:

$$x\otimes y = \boldsymbol{g_{\underline{\kappa}+\underline{\kappa}',\underline{\lambda}+\underline{\lambda}'}}, \quad \forall x\in\boldsymbol{g_{\underline{\kappa},\underline{\lambda}}},\ y\in\boldsymbol{g_{\underline{\kappa}',\underline{\lambda}'}},\ \underline{\kappa}, \underline{\kappa}'\in Z_s,\ \underline{\lambda}, \underline{\lambda}'\in Z_r$$

Then the hyperstructure $(\mathbf{P_n},\otimes)$ is an H_v-group where its fundamental group is isomorphic to the $(\boldsymbol{Z_s}\times\boldsymbol{Z_r}, +)$.

Proof. It is clear that the only case we have β*-equivalent elements are when we have elements from $\boldsymbol{g}_{\underline{\kappa},\underline{1}}$. Therefore, from the reproductivity, the only β*-equivalent classes are the $\boldsymbol{g}_{\underline{\kappa},\underline{\lambda}}$, $\underline{\kappa}\in Z_s$, $\underline{\lambda}\in Z_r$. That proves that

$$(\mathbf{P_n},\otimes)/\beta^* \approx (\boldsymbol{Z}_s\times\boldsymbol{Z}_r, +)/\beta^*. \quad \blacksquare$$

We can generalize the LV-construction III by considering any group or H_v-group instead of above used Z_n ones, in an analogous way.

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