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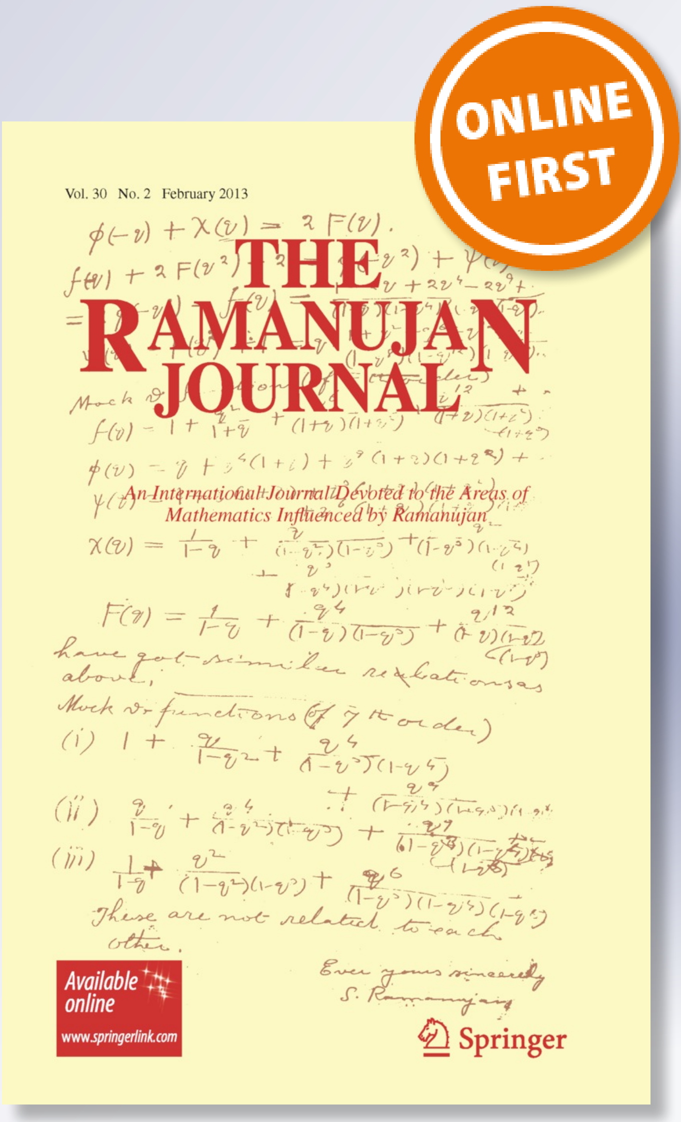

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Abstract We study the odd prime values of the Ramanujan tau function, which form a thin set of large primes. To this end, we define $LR(p,n) := \tau(p^{n-1})$ and we show that the odd prime values are of the form $LR(p,q)$ where p, q are odd primes. Then we exhibit arithmetical properties and congruences of the LR numbers using more general results on Lucas sequences. Finally, we propose estimations and discuss numerical results on pairs (p,q) for which $LR(p,q)$ is prime.

Keywords Ramanujan function · Primality · Lucas sequences

Mathematics Subject Classification (2010) 11A41 · 11F30 · 11Y11

1 Introduction

The tau function is defined as the Fourier coefficients of the modular discriminant

$$\Delta(z) = q \prod_{n=1}^{+\infty} (1 - q^n)^{24} = \sum_{n=1}^{+\infty} \tau(n) q^n,$$

where z lies in the complex upper half-plane and $q = e^{2\pi i z}$.

Nearly a century ago, the Indian mathematician Srinivasa Ramanujan showed great interest in the tau function and discovered some of its remarkable properties.

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Below are listed the known identities and congruences for the tau function:

$$\tau(nm) = \tau(n)\tau(m) \qquad \text{for } n, m \text{ coprime integers;} \tag{1}$$

$$\tau\left(p^{r+1}\right) = \tau(p)\tau\left(p^r\right) - p^{11}\tau\left(p^{r-1}\right) \qquad \text{for } p \text{ prime and } r \text{ an integer } \geq 1; \tag{2}$$

$$\tau(n) \equiv \sigma_{11}(n) \left(\text{mod } 2^{11}\right) \qquad \text{for } n \equiv 1 \text{ (mod 8);} \tag{3}$$

$$\tau(n) \equiv 1217\sigma_{11}(n) \left(\text{mod } 2^{13}\right) \qquad \text{for } n \equiv 3 \text{ (mod 8);} \tag{4}$$

$$\tau(n) \equiv 1537\sigma_{11}(n) \left(\text{mod } 2^{12}\right) \qquad \text{for } n \equiv 5 \text{ (mod 8);} \tag{5}$$

$$\tau(n) \equiv 705\sigma_{11}(n) \left(\text{mod } 2^{14}\right) \qquad \text{for } n \equiv 7 \text{ (mod 8);} \tag{6}$$

$$\tau(n) \equiv n^{-610}\sigma_{1231}(n) \left(\text{mod } 3^{6}\right) \qquad \text{for } n \equiv 1 \text{ (mod 3);} \tag{7}$$

$$\tau(n) \equiv n^{-610}\sigma_{1231}(n) \left(\text{mod } 3^{7}\right) \qquad \text{for } n \equiv 2 \text{ (mod 3);} \tag{8}$$

$$\tau(n) \equiv n^{-30}\sigma_{71}(n) \left(\text{mod } 5^{3}\right) \qquad \text{for } n \not\equiv 0 \text{ (mod 5);} \tag{9}$$

$$\tau(n) \equiv n\sigma_{9}(n) \text{ (mod 7)} \qquad \text{for } n \equiv 0, 1, 2, 4 \text{ (mod 7);} \tag{10}$$

$$\tau(n) \equiv n\sigma_{9}(n) \left(\text{mod } 7^{2}\right) \qquad \text{for } n \equiv 3, 5, 6 \text{ (mod 7);} \tag{11}$$

$$\tau(p) \equiv 0 \text{ (mod 23)} \qquad \text{for } p \text{ prime, } \left(\frac{p}{23}\right) = -1; \tag{12}$$

$$\tau(p) \equiv \sigma_{11}(p) \left(\text{mod } 23^{2}\right) \qquad \text{for } p \text{ prime of the form } u^2 + 23v^2; \tag{13}$$

$$\tau(p) \equiv -1 \text{ (mod 23)} \qquad \text{for other prime } p; \tag{14}$$

$$\tau(n) \equiv \sigma_{11}(n) \text{ (mod 691)}, \tag{15}$$

where u, v are integers, $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n, and $(\frac{\cdot}{\cdot})$ denotes the Legendre symbol.

All congruences are given with their respective authors in [11], except (13) which is due to Serre. Swinnerton-Dyer showed that there are no congruences for $\tau(n)$ modulo primes other than 2, 3, 5, 7, 23 and 691.

Ramanujan [7] conjectured, and Deligne proved, the upper bound

$$\left|\tau(p)\right| \leq 2p^{\frac{11}{2}} \qquad \text{for } p \text{ prime.} \tag{16}$$

We recall that the values of the tau function are almost always divisible by any integer [10, p. 243].

In this paper, we will study the integers n for which $\tau(n)$ is an odd prime, disregarding the sign of $\tau(n)$. It is easily seen that $\tau(n)$ is odd if and only if n is an odd square. Then from (1) one should expect the smallest integer n for which $\tau(n)$ is an odd prime to be of the form p^r where r is even and p odd prime.

Indeed, D. H. Lehmer [2] found that $n = 63001 = 251^2$ is the smallest integer for which $\tau(n)$ is prime:

$$\tau\left(251^2\right) = -80561663527802406257321747.$$

Without the power of today's computers, proving such a result was not straightforward.

2 LR numbers

We propose to define the *LR* family of integers, in memory of D.H. Lehmer and S. Ramanujan, as follows:

Definition 1 Let p, q be odd primes. Then we define $LR(p,q) := \tau(p^{q-1})$. More generally, we shall use the notation $LR(p,n) := \tau(p^{n-1})$ for all positive integers n and we set the value $LR(p,0) := 0$.

The main motivation for the introduction of the previous definition is related to Theorem 1, for which we will give a proof. It states a strong necessary condition on the integers n such that $\tau(n)$ is an odd prime. Our notation will prove to be relevant as the "diagonal" terms $LR(p,p)$ have specific arithmetical properties (see Theorem 4).

In what follows, a prime p such that $p \nmid \tau(p)$ is called ordinary. Otherwise p is said to be non-ordinary.

Theorem 1 *Let n be a positive integer such that $\tau(n)$ is an odd prime. Then $n = p^{q-1}$ where p and q are odd primes and p is ordinary.*

Remark 1 Only finitely many non-ordinary primes are known to exist: 2, 3, 5, 7, 2411, 7758337633. We expect them to be the smallest elements of a very thin infinite set [4]. They are also referred to as supersingular primes.

Now we provide several formulations, notations, and intermediate results concerning the *LR* numbers that will be useful in our study.

Let p be an odd prime. The recurrence relation (2) implies that $LR(p,n)$ is the nth term of the Lucas [3] sequence associated with the polynomial $X^2 - \tau(p)X + p^{11}$. Hence for $n > 0$, $LR(p,n)$ is a polynomial of degree $(n-1)$ in $\tau(p)$ and p^{11}:

$$LR(p,n) = \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \binom{n-1-k}{k} p^{11k} \tau(p)^{n-1-2k}. \tag{17}$$

The divisibility property of the Lucas sequences applies:

$$\text{if} \quad m \mid n, \quad \text{then} \quad LR(p,m) \mid LR(p,n). \tag{18}$$

Our sequence has discriminant

$$D_p := \tau(p)^2 - 4p^{11} \tag{19}$$

which is negative by (16). We get the general expression

$$LR(p,n) = \frac{\alpha_p^n - \overline{\alpha}_p{}^n}{\alpha_p - \overline{\alpha}_p}, \quad \text{with } \alpha_p = \frac{\tau(p) + \sqrt{D_p}}{2}.$$

Table 1 First values of $\tau(p)$, θ_p/π, $\cos\theta_p$ and $\sin\theta_p$

p	$\tau(p)$	θ_p/π	$\cos\theta_p$	$\sin\theta_p$
2	−24	0.585426	−0.265165	0.964203
3	252	0.403225	0.299367	0.954138
5	4830	0.387673	0.345607	0.938379
7	−16744	0.560289	−0.188274	0.982117
11	534612	0.333173	0.500436	0.865773
13	−577738	0.569230	−0.215781	0.976442
17	−6905934	0.700803	−0.589825	0.807531
19	10661420	0.335570	0.493901	0.869518
23	18643272	0.402350	0.301988	0.953312
29	128406630	0.302561	0.581257	0.813720
31	−52843168	0.553006	−0.165756	0.986167
37	−182213314	0.569299	−0.215993	0.976395
41	308120442	0.433411	0.207673	0.978198
43	−17125708	0.502827	−0.008882	0.999961
47	2687348496	0.173811	0.854586	0.519310

Following Ramanujan [7], we define the angles $\theta_p \in (0,\pi)$ such that $\tau(p) = 2p^{\frac{11}{2}}\cos\theta_p$. Some values of $\tau(p)$ and θ_p/π are listed in Table 1. Then we derive an equivalent formulation related to the Chebyshev polynomials of the second kind:

$$LR(p,n) = p^{\frac{11(n-1)}{2}}\frac{\sin(n\theta_p)}{\sin\theta_p} = \prod_{k=1}^{n-1}\left(\tau(p) - 2p^{\frac{11}{2}}\cos\frac{k\pi}{n}\right). \tag{20}$$

Hence, a fair estimation of the size of $|LR(p,n)|$ is given by $p^{\frac{11}{2}(n-1)}$ in most cases. This is supported by the numerical results.

Theorem 2, due to Murty, Murty and Shorey [6], proves that the tau function takes any fixed odd integer value, and *a fortiori* any odd prime value, finitely many times.

Theorem 2 *There exists an effectively computable absolute constant $c > 0$, such that for all positive integers n for which $\tau(n)$ is odd, we have*

$$\left|\tau(n)\right| \geq (\log n)^c.$$

The next result is somehow related to Theorem 2 (see Remark 2), and will be used in the proof of Theorem 1.

Lemma 1 *The equation $\tau(n) = \pm 1$ has no solution for $n > 1$.*

Proof (sketch) By property (1), we can assume without loss of generality that $n = p^r$ for a prime p and integer $r > 0$. Thus $\tau(n) = LR(p, r+1)$.

Now it suffices to apply known results on Lucas sequences (Theorems C, 1.3, and 1.4 in [1]) to show that $LR(p, r+1)$ has a primitive divisor. □

Remark 2 It is not quite obvious for us if Lemma 1 is a corollary of Theorem 2, as the latter appears to be essentially of qualitative nature. Moreover, the effectiveness in the special case $\tau(n) = \pm 1$ is nowhere mentioned in [6].

3 Proof of Theorem 1

Let n be a positive integer such that $\tau(n)$ is an odd prime.

It follows from the multiplicative property (1) and Lemma 1 that n is a power of a prime p. Thus $n = p^r$ for some positive integer r and $\tau(n) = LR(p, r+1)$.

From the divisibility property (18) and once again Lemma 1, it turns out that $r + 1 = q$ where q is prime. Since $LR(p, 2)$ is even, we have $r > 1$ and $q \neq 2$.

Now suppose that p is non-ordinary. We get $p|\tau(p)$ which in turn implies that $p^2|\tau(n)$ by (17). Therefore, we have reached a contradiction.

4 Arithmetical properties

The theory of Lucas sequences is well developed (see, e.g., [9]) and has many implications for the LR numbers. It leads to the arithmetical properties:

$$\gcd\big(LR(p,m), LR(p,n)\big) = LR\big(p, \gcd(m,n)\big); \tag{21}$$

$$LR(p,q) \equiv \left(\frac{D_p}{q}\right) \pmod q; \tag{22}$$

$$\text{if} \quad q \nmid p \cdot \tau(p), \quad \text{then} \quad q|LR\left(p, q - \left(\frac{D_p}{q}\right)\right); \tag{23}$$

$$LR(p, 2n+1) = LR(p, n+1)^2 - p^{11} LR(p,n)^2 \tag{24}$$

for m, n two positive integers and p, q two odd primes. The discriminant D_p is defined by (19).

Theorems 3 and 4 will prove to be useful in our estimations and numerical calculations (see Sects. 6, 7 and Appendix). As they are also related to known properties of the Lucas sequences, we only give sketches of proof.

Theorem 3 *Let p and q be two odd primes, p ordinary.*
If d is a prime divisor of $LR(p,q)$, then $d \equiv \pm 1 \pmod{2q}$ or $d = q$.
Moreover, $q|LR(p,q)$ if and only if $q|D_p$.

Proof (sketch) We consider the number $LR(p, \gcd(q, d - (\frac{D_p}{d})))$ and we apply successively (21) and (23). Then we get

$$\gcd\left(q, d - \left(\frac{D_p}{d}\right)\right) \neq 1,$$

and the theorem follows by (22). □

Theorem 4 *Let p be an ordinary odd prime.*

If $p \equiv 1 \pmod 4$, then $LR(p,p)$ is composite.

If $p \equiv 3 \pmod 4$ and d is a prime divisor of $LR(p,p)$, then $d \equiv \pm 1 \pmod{4p}$.

Proof (sketch) The first part of the theorem follows from formulation (20) leading to the generic factorization $LR(p,p) = N_0 N_1$ where

$$N_j = \prod_{k=1}^{\frac{p-1}{2}} \left(\tau(p) - (-1)^{j+k} \left(\frac{2k}{p} \right) 2p^{\frac{11}{2}} \cos \frac{k\pi}{p} \right), \quad \text{for } j = 0, 1.$$

One may verify that N_0 and N_1 are integers using the Gauss sum value

$$\sum_{k=1}^{p-1} \left(\frac{k}{p} \right) e^{\frac{2ki\pi}{p}} = \sqrt{p}, \quad \text{for } p \equiv 1 \pmod 4.$$

For the second part, we apply (24) and obtain that p is a quadratic residue modulo d. The desired result easily follows by combining the law of quadratic reciprocity with the congruence $d \equiv \pm 1 \pmod{2p}$. □

5 Congruences modulo $p \pm 1$

No congruence modulo p is known for $\tau(p)$ (see, e.g., [4]), hence *a fortiori* for the numbers $LR(p,n)$. Here we briefly study the sets $\mathcal{P}^+$ and $\mathcal{P}^-$ of primes p for which the numbers $LR(p,n)$ have elementary congruences modulo $p+1$ and $p-1$, respectively, as specified in Lemma 2.

Lemma 2 *Let $\mathcal{P}^+$ be the set of odd primes p such that $\tau(p) \equiv 0 \pmod{p+1}$. Let $p \in \mathcal{P}^+$. Then $LR(p,n) \equiv 0 \pmod{p+1}$ if n is even, and $LR(p,n) \equiv 1 \pmod{p+1}$ if n is odd.*

Let $\mathcal{P}^-$ be the set of odd primes p such that $\tau(p) \equiv 2 \pmod{p-1}$. Let $p \in \mathcal{P}^-$. Then $LR(p,n) \equiv n \pmod{p-1}$ for all positive integer n.

Proof (sketch) The recurrence relation (2) leads to an easy proof by induction for all positive integers n. □

Lemma 3 states that $\mathcal{P}^+$ and $\mathcal{P}^-$ both include most of the small primes. Nevertheless, there is numerical evidence that they have density zero among the primes.

Lemma 3 *Let $A := 2^{14} \cdot 3^7 \cdot 5^3 \cdot 7^2 \cdot 23 \cdot 691$, and let $\mathcal{P}_0^+$ be the set of odd primes p such that $p+1 | A$. Then $\mathcal{P}_0^+ \subset \mathcal{P}^+$.*

Let $B := 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 691$, and let $\mathcal{P}_0^-$ be the set of odd primes p such that $p-1 | B$. Then $\mathcal{P}_0^- \subset \mathcal{P}^-$.

Proof (sketch) Let $p \in \mathcal{P}_0^+$. Then $p = 2^{r(2)} \cdot 3^{r(3)} \cdot 5^{r(5)} \cdot 7^{r(7)} \cdot 23^{r(23)} \cdot 691^{r(691)} - 1$, with $r(q) \leq 14, 7, 3, 2, 1, 1$ for $q = 2, 3, 5, 7, 23, 691$, respectively.

From (3), (4), (5), (6), (8), (9), (11), (12), and (15), it follows that $\tau(p) \equiv 0 \pmod{q^{r(q)}}$ for $q = 2, 3, 5, 7, 23, 691$. Thus $\tau(p) \equiv 0 \pmod{p+1}$, that is $p \in \mathcal{P}^+$.

Similarly, we prove that $\mathcal{P}_0^- \subset \mathcal{P}^-$ using (3), (4), (5), (6), (7), (9), (10), and (15). □

Remark 3 It is not possible to increase any of the exponents in the previous definitions of A and B without finding counter-examples of Lemma 3.

Obviously, $\mathcal{P}_0^+$ and $\mathcal{P}_0^-$ are finite sets. They comprise respectively 1140 and 325 primes and their smallest and largest elements are given below:

- $\mathcal{P}_0^+ = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 47, 53, 59, 71, 79, 83, 89, 97, \ldots, \frac{1}{6}A - 1, \frac{1}{5}A - 1\}$ and $\max \mathcal{P}_0^+ \approx 6.97 \times 10^{14}$;
- $\mathcal{P}_0^- = \{3, 5, 7, 11, 13, 17, 19, 29, 31, 37, 41, 43, 61, 71, 73, 97, \ldots, \frac{1}{32}B + 1, B + 1\}$ and $\max \mathcal{P}_0^- \approx 9.02 \times 10^{11}$.

Now if we define the residual sets $\mathcal{P}_1^+ := \mathcal{P}^+ \setminus \mathcal{P}_0^+$ and $\mathcal{P}_1^- := \mathcal{P}^- \setminus \mathcal{P}_0^-$, it turns out that

- $\mathcal{P}_1^+ = \{593, 1367, 2029, 2753, 4079, 4283, 7499, 7883, 9749, 11549 \ldots\}$;
- $\mathcal{P}_1^- = \{103, 311, 691, 829, 1151, 1373, 2089, 2113, 2411, 2647, \ldots\}$.

Note that

- $\mathcal{P}^+$ contains the Mersenne primes $M_p := 2^p - 1$ for $p = 2, 3, 5, 7, 13, 17$, and 19, but M_{31} is not in $\mathcal{P}^+$;
- $\mathcal{P}^-$ contains all known Fermat primes $F_n := 2^{2^n} + 1$ for $n = 0, 1, 2, 3, 4$;
- $\mathcal{P}_1^-$ contains the non-ordinary prime 2411 (see Remark 1), but the next one, 7758337633, is not in $\mathcal{P}^+ \cup \mathcal{P}^-$.

6 Estimations

Here we provide various estimates on the number and distribution of *LR* primes with only little justification. Then it will be compared with numerical results.

We refer to Wagstaff's heuristic reasoning [12] about the probability for a Mersenne number M_p to be prime, mainly considering that all divisors are of the form $2kp + 1$. The proposed value is

$$\frac{e^\gamma \log 2p}{p \log 2},$$

where $\gamma = 0.577215\ldots$ is Euler's constant.

Let p and q be two odd primes such that $p \neq q$ and $\tau(p) \not\equiv 0 \pmod{p}$. We know from Theorem 3 that all prime divisors of $LR(p, q)$ are of the form $2kq \pm 1$, except possibly q with probability $P(q)$. The expected value of $P(q)$ is roughly $1/q$, unless $q = 3, 5, 7$, or 23 for which we have the congruences (7) to (14). We easily get the exceptional values of $P(q)$ for $q = 3, 5$ or 7 by considering all residues of p modulo q (see also [10, 11]), whereas $P(23)$ is the proportion of primes of the form $u^2 + 23v^2$:

$$P(3) = \frac{1}{2}; \qquad P(5) = \frac{1}{4}; \qquad P(7) = \frac{1}{2}; \qquad P(23) = \frac{1}{6}.$$

Now we estimate the probability that $LR(p, q)$ is prime by

$$\frac{e^\gamma \log 2q}{\log |LR(p,q)|}\big(1 - P(q)\big) \approx \frac{2e^\gamma \log 2q}{11(q-1)\log p}\big(1 - P(q)\big). \tag{25}$$

In the general case where $q \neq 3, 5, 7, 23$, we have $P(q) \approx 1/q$ and (25) simplifies to

$$\frac{2e^\gamma \log 2q}{11q \log p}.$$

If we assume further that $p = q$ and $p \equiv 3 \pmod 4$, then the same reasoning, using the results from Theorem 4, leads to the probability

$$\frac{e^\gamma \log 4p}{\log |LR(p,p)|} \approx \frac{2e^\gamma \log 4p}{11(p-1)\log p}. \tag{26}$$

Now we consider two large integers $p_{\max} \gg 1$ and $q_{\max} \gg 1$. The expected number of primes of the form $LR(p, q)$ for prime p fixed and $q < q_{\max}$ is

$$\frac{2e^\gamma}{11 \log p} \sum_{\text{odd prime } q < q_{\max}} \frac{\log 2q}{q} \sim \frac{2e^\gamma \log q_{\max}}{11 \log p}. \tag{27}$$

Therefore, our estimate at first order for the number of primes of the form $LR(p, q)$ with $p < p_{\max}$ and $q < q_{\max}$ is

$$\frac{2e^\gamma \log q_{\max}}{11} \sum_{\text{ordinary prime } p < p_{\max}} \frac{1}{\log p} \sim \frac{2e^\gamma p_{\max} \log q_{\max}}{11(\log p_{\max})^2}. \tag{28}$$

Using (26), we also estimate the number of primes of the form $LR(p, p)$ with $p < p_{\max}$ by

$$\frac{2e^\gamma}{11} \sum_{\substack{\text{ordinary prime } p < p_{\max} \\ p \equiv 3 \pmod 4}} \frac{\log 4p}{(p-1)\log p} \sim \frac{e^\gamma}{11} \log\log p_{\max}. \tag{29}$$

7 Numerical results

We have checked the (probable) primality of $LR(p, q)$ for all pairs (p, q) of odd primes in Table 2 (see Appendix for details). The estimates (*) follow from our first order approximations (28) and (29), whereas the other estimates (**) are simply a sum of the related expressions (25) or (26) over all considered p, q values. The latter

Table 2 Counting the primes and PRP's of the form $LR(p,q)$

Conditions on p, q	Max. number of digits	Number of (probable) primes		
		Actual	Expected*	Expected**
$(p < 10^6)$ and $(q < 100)$	3169	7312	7813	7203
$(p < 20000)$ and $(q < 1000)$	23560	491	456	520
$(p < 1000)$ and $(q < 5000)$	82432	76	57.8	74.7
$(p < 300)$ and $(q < 20000)$	271302	32	29.6	38.9
$(p < 100)$ and $(q < 30000)$	327687	17	15.7	18.3
$p = q < 20000$	472856	1	0.37	0.32

estimates are in good agreement with the numerical results when the number of *LR* primes is significant.

In Table 3, we give a list of 81 pairs (p,q) of odd primes such that $p < 1000$ and $LR(p,q)$ is prime or probable prime (PRP). The number of decimal digits is ranging from 26 to 250924. The largest known prime value of the tau function is $LR(157, 2207)$, thanks to F. Morain (see Appendix). So far, $LR(41, 28289)$ is the largest known PRP value.

By estimation (27), we expect the existence of infinitely many primes of the form $LR(p,q)$ for each ordinary prime p. However, we found no PRP for $p = 13, 19, 23, 31, 37, 43, 53, 61, 67, 71, 73, 83, \ldots$.

Remark 4 Considering the list $p = 11, 17, 29, 41, 47, 59, 79, 89, 97, \ldots$ for which we know LR (probable) primes, it is remarkable that the six first values correspond exactly to the odd values in the sequence of the Ramanujan primes: 2, 11, 17, 29, 41, 47, 59, 67, 71, 97, ... This sequence was introduced to provide a short proof of Bertrand's postulate [8]. However, we found no significant correlation past the value 59 and do not expect a close mathematical relationship between the tau function and the Ramanujan primes.

Note that the only prime of the form $LR(p,p)$ for $p < 20000$ is $LR(47,47)$. Our estimation (29) suggests the existence of infinitely many such values.

We give in Table 4 the number of PRP's of the form $LR(p,q)$, for each given odd prime $q < 100$ and all $p < 10^6$, along with the estimates (**) obtained by summing over prime p the expression (25). We partly explain the discrepancy between the actual data and our expectations by considering the compositeness of some $2kq \pm 1$ numbers for small positive integers k. For example, if $q = 59$, then those numbers are composite for $k = 1, 2, 4, 5, 8, \ldots$, which in turn implies that $LR(p, 59)$ has no divisor smaller than 353, except possibly p and 59. In this case, we observe that the number of primes is effectively higher than expected.

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Table 3 Known pairs (p, q), $p < 1000$, such that $LR(p, q)$ is prime (P) or probable prime (PRP). ECPP (◇) method has been used for the primality of two large values

p	q	Digits	Primality	p	q	Digits	Primality
11	317	1810	P	439	29	407	P
17	433	2924	P	449	547	7965	PRP
29	31	242	P	461	3019	44215	PRP
29	83	660	P	463	2753	40347	PRP
29	229	1834	P	487	479	7066	PRP
41	2297	20367	PRP	491	167	2457	P
41	28289	250924	PRP	503	73	1070	P
47	5	37	P	557	109	1631	P
47	47	424	P	571	1091	16526	PRP
47	4177	38404	PRP	587	1093	16629	PRP
59	1381	13441	PRP	607	13	184	P
59	8971	87365	PRP	613	47	706	P
79	1571	16386	PRP	613	1013	15515	PRP
79	6317	65920	PRP	619	1297	19900	PRP
89	73	772	P	643	953	14703	P ◇
97	331	3606	P	673	1019	15834	PRP
97	887	9682	PRP	677	3	32	P
103	14939	165374	PRP	691	1523	23770	PRP
109	373	4169	PRP	739	2503	39475	PRP
113	197	2214	P	761	13	190	P
157	2207	26643	P ◇	773	67	1049	P
173	103	1256	P	787	73	1147	P
197	5	50	P	809	149	2367	P
199	4519	57125	PRP	811	43	671	P
223	101	1292	P	821	1163	18626	PRP
223	281	3617	P	829	11	161	P
223	9431	121795	PRP	839	4177	67153	PRP
227	11	130	P	857	683	11002	PRP
239	107	1387	P	857	3847	62042	PRP
251	3	26	P	877	3617	58531	PRP
251	1193	15733	PRP	881	241	3888	PRP
257	1699	22506	PRP	881	251	4050	PRP
281	19	243	P	937	59	949	P
331	2129	29492	PRP	941	349	5692	PRP
349	409	5706	PRP	947	41	654	P
353	239	3335	P	953	557	9111	PRP
379	11	142	P	971	3	33	P
401	59	831	P	971	433	7098	PRP
409	4423	63520	PRP	977	59	954	P
421	89	1271	P	983	3	33	P
421	317	4561	PRP				

Table 4 For odd primes $q < 100$, actual and expected** number of primes $p < 10^6$ such that $LR(p, q)$ is PRP, and first values of p

q	Actual	Expected**	p
3	838	904	251, 677, 971, 983, 1229, ...
5	910	871	47, 197, 1123, 2953, 3373, ...
7	438	444	1151, 2141, 5087, 6907, 7129, ...
11	663	567	227, 379, 829, 1217, 1367, ...
13	593	506	607, 761, 1033, 1867, 1999, ...
17	438	419	1301, 1319, 1373, 8363, 9209, ...
19	321	386	281, 4751, 5717, 7103, 10181, ...
23	273	293	1013, 2113, 6577, 6581, 8609, ...
29	263	283	439, 1783, 3109, 3209, 3301, ...
31	256	269	29, 6737, 7757, 8243, 8707, ...
37	208	235	1061, 1217, 1621, 2699, 3167, ...
41	214	217	947, 2671, 4817, 5231, 6079, ...
43	242	209	811, 7549, 8089, 9337, 9923, ...
47	232	195	47, 613, 1361, 2963, 4219, ...
53	143	178	4153, 4457, 6311, 23209, 30211, ...
59	232	163	401, 937, 977, 1609, 3121, ...
61	181	159	1583, 1747, 5209, 7057, 10079, ...
67	159	148	773, 1597, 2969, 3823, 4603, ...
71	142	141	1601, 6469, 10037, 15391, 23371, ...
73	144	138	89, 503, 787, 7687, 12689, ...
79	104	129	21193, 23339, 31847, 38239, 38327, ...
83	112	124	29, 2927, 3391, 7873, 8597, ...
89	104	117	421, 2843, 4637, 4937, 5659, ...
97	102	110	5261, 7537, 11933, 22613, 23627, ...

Appendix: Computations

Here we provide a PARI/GP implementation of the LR numbers, using a known formula (see, e.g., [4]) along with the recurrence relation (2).

```
LR(p,n)={
  local(j,p11,s10,t,tp,t0,t1,t2,tmax);
  tmax=floor(2*sqrt(p));
  s10=sum(t=1, tmax, (t^10)*qfbhclassno(4*p-t*t));
  tp=(p+1)*(42*p^5-42*p^4-48*p^3-27*p^2-8*p-1)-s10;
  t0=1; t1=tp; p11=p^11;
  for (j=1, n-2, t2=tp*t1-p11*t0; t0=t1; t1=t2);
  if (n==1, t1=1);
  return(t1)
}
```

We took about seven months of numerical investigations for primes of the form $LR(p, q)$, p and q odd primes, using the multiprecision software PARI/GP (version 2.3.5) and PFGW (version 3.4.5) through four stages:

1. Finding small divisors of the form $2kq \pm 1$ with PARI/GP;
2. 3-PRP tests with PFGW;
3. APRCL primality tests for all PRP's up to 3700 decimal digits with PARI/GP;
4. Baillie-PSW PRP tests for all PRP's above 3700 decimal digits with PARI/GP.

Stage 4 leads to a greater probability of primality than stage 2 (there is no known composite number which is passing this test), but takes more time.

We point out that François Morain provides primality certificates for two large *LR* numbers (see diamonds $\diamond$ in Table 3) on his web page. He used his own software fastECPP, implementing a fast algorithm of elliptic curve primality proving [5], on a computer cluster. His calculations required respectively 355 and 2355 days of total CPU time, between January and April 2011. Since $LR(157, 2207)$ has 26643 decimal digits, it appears to be the largest prime certification using a general-purpose algorithm, at the date of submission.

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