# The Hypergroups of order 3

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# Definitions

- **Definition :** A hypergroupoid H=<H,*> is a non-empty set H equipped with a hyperoperation *. (i.e. a mapping whose domain is HxH and whose range is p(H) )

- **Definition** : A hypergroupoid is called a hypergroup if the hyperoperation satisfies the following axioms :
  - $(x*y)*z = x*(y*z)$ for all x,y,z in H *(associativity)*
  - $x*H = H*x = H$ for all x in H *(reproduction)*

- **Definition :** Let G=<G,.> be a group and P a non-empty subset of G, a P-hypergroup is the hypergroup $G_P$=<G,$*_P$> equipped with the following hyperoperation

$$*_P : (x,y) \longrightarrow x.P.y$$

# Definitions

- **Definition :** A hypergroup H=<H,*> is called cyclic with finite period with respect an element h of H if there exists an integer v such that $H=h^1 \cup h^2 \cup \ldots . \cup h^v$.

- **Definition** : A hypergroupoid is called a hyperstructure if the hyperoperation satisfies the following axioms :
  - $(x*y)*z \cap x*(y*z) \neq \phi$ for all x,y,z in H *(weak associativity)*
  - $x*H = H*x = H$ for all x in H *(reproduction)*

# Posets, Groups and Hypergroups

**Theorem (2004)**

A poset Po can be associated to every P-hypergroup.

**Theorem (2004)**

A poset Po of cardinality $\mathbf{a^2+a}$ can be associated to every <G,P*> P-hypergroup with G of cardinality **a**.

# Posets, Groups and Hypergroups

**Theorem (2004)**
Let **a** a prime number, <G,P*> P-hypergroup and G a group of cardinality **a**=2 or 3,5,7 or 11 and more then there is an associated poset Po of cardinality respectively **a** or **3a** or **2a** which has an automorphism group of order **a.**

**Theorem (2004)**
Let <G,P*> P-hypergroup and G a finite group of cardinal **a**, non direct product of two groups, generated by elements which are two by two of distinct order then there exists a poset which automorphisms group is isomorphic to G and with cardinal is equal to **3a**.

# The Hypergroups of order 2

- 81 possible configurations
- 35 candidates verifying axiom of reproduction
- 30 candidates verifying associativity
- 14 hypergroups

**Theorem (Vougiouklis 1980)**
There are 8 non isomorphic hypergroups of order 2.

# The Hypergroups of order 2

| $H_1 \cong \mathbf{Z}_2$ | |
|---|---|
| a | b |
| b | a |

| $H_2$ | |
|---|---|
| a | b |
| b | {a,b} |

| $H_3$ | |
|---|---|
| a | b |
| {a,b} | {a,b} |

| $H_4$ | |
|---|---|
| a | {a,b} |
| b | {a,b} |

| $H_5$ | |
|---|---|
| a | {a,b} |
| {a,b} | b |

| $H_6$ | |
|---|---|
| a | {a,b} |
| {a,b} | {a,b} |

| $H_7$ | |
|---|---|
| b | {a,b} |
| {a,b} | {a,b} |

| $H_8$ | |
|---|---|
| {a,b} | {a,b} |
| {a,b} | {a,b} |

| $H_i$ | $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ | $H_6$ | $H_7$ | $H_8$ |
|---|---|---|---|---|---|---|---|---|
| $\|Aut(H_i)\|$ | 2 | 2 | 2 | 2 | 1 | 2 | 2 | 1 |

The hypergroups of order 3

# The Hypergroups of order 3

- 40353607 possible configurations
- 10323979 candidates verifying axiom of reproduction
- 28111 candidates verifying associativity
- 23192 hypergroups

> **Theorem (2004)**
> There are 3999 non isomorphic hypergroups of order 3.

# Classification

<table>
<tr><td colspan="2">CYCLIC & ABELIAN<br><br>437</td><td colspan="2">CYCLIC & NON ABELIAN<br><br>3439</td></tr>
<tr><td colspan="2">NON CYCLIC & ABELIAN</td><td colspan="2">NON CYCLIC & NON ABELIAN</td></tr>
<tr><td>PROJECTIVE<br>7</td><td>NON PROJECTIVE<br>22</td><td>PROJECTIVE<br>29</td><td>NON PROJECTIVE<br>65</td></tr>
</table>

The hypergroups of order 3

# Automorphism groups

| | | | | |
|---|---|---|---|---|
| Order of automorphism group | 1 | 2 | 3 | 6 |
| Number of hypergroups | 6 | 10 | 244 | 3739 |

# Rigid Hypergroups

**Proposition**
There are only 2 rigid hypergoups of order 2.

**Theorem**
There are only 6 rigid hypergoups of order 3.

# Rigid hypergroups

- 6 rigid hypergroups of order 3

| | | |
|---|---|---|
| a | {a,b} | {a,c} |
| {a,b} | b | {b,c} |
| {a,c} | {b,c} | c |

| | | |
|---|---|---|
| a | {a,b,c} | {a,b,c} |
| {a,b,c} | b | {a,b,c} |
| {a,b,c} | {a,b,c} | c |

| | | |
|---|---|---|
| {b,c} | {a,b} | {a,c} |
| {a,b} | {a,c} | {b,c} |
| {a,c} | {b,c} | {a,b} |

| | | |
|---|---|---|
| {b,c} | {a,b,c} | {a,b,c} |
| {a,b,c} | {a,c} | {a,b,c} |
| {a,b,c} | {a,b,c} | {a,b} |

| | | |
|---|---|---|
| {a,b,c} | {a,b} | {a,c} |
| {a,b} | {a,b,c} | {b,c} |
| {a,c} | {b,c} | {a,b,c} |

| | | |
|---|---|---|
| {a,b,c} | {a,b,c} | {a,b,c} |
| {a,b,c} | {a,b,c} | {a,b,c} |
| {a,b,c} | {a,b,c} | {a,b,c} |

# Hypergroups with scalar unit

- 40353607 possible configurations
- 10323979 candidates verifying axiom of reproduction
- 28111 candidates verifying associativity
- 198 hypergroups with scalar unit

**Theorem (2004)**

There are 37 non isomorphic hypergroups of order 3 with scalar unit.

# Hyperstructures of order 2

- 81 possible configurations
- 65 candidates verifying weak associativity
- 35 candidates verifying axiom of reproduction
- 35 hyperstructures

**Proposition**

If <H,*> is a hypergroupoid of order 2 verifying axiom of reproduction then <H,*> is a hyperstructure.

**Theorem (2004)**

There are 20 non isomorphic hyperstructures of order 2.

# Hyperstructures of order 2

- (a*a,a*b,b*a,b*b)

| | |
|---|---|
| (a, b, b, a) | (b, a, {a,b}, {a,b}) |
| (a, b, b, {a,b}) | (b, {a,b}, a, {a,b}) |
| (a, b, {a,b}, a) | (b, {a,b}, {a,b}, a) |
| (a, b, {a,b}, {a,b}) | (b, {a,b}, {a,b}, {a,b}) |
| (a, {a,b}, b, a) | ({a,b}, a, a, {a,b}) |
| (a, {a,b}, b, {a,b}) | ({a,b}, a, b, {a,b}) |
| (a, {a,b}, {a,b}, a) | ({a,b}, a, {a,b}, {a,b}) |
| (a, {a,b}, {a,b}, b) | ({a,b}, b, a, {a,b}) |
| (a, {a,b}, {a,b}, {a,b}) | ({a,b}, b, {a,b}, {a,b}) |
| (b, a, a, {a,b}) | ({a,b}, {a,b}, {a,b}, {a,b}) |

The hypergroups of order 3

# Hyperstructures of order 2

- Rigid hyperstructures

| $H'_{13}$ | |
|---|---|
| b | {a,b} |
| {a,b} | a |

| $H'_{16}$ | |
|---|---|
| {a,b} | a |
| b | {a,b} |

| $H'_{18}$ | |
|---|---|
| {a,b} | b |
| a | {a,b} |

| $H_5$ | |
|---|---|
| a | {a,b} |
| {a,b} | b |

| $H_8$ | |
|---|---|
| {a,b} | {a,b} |
| {a,b} | {a,b} |

# Hyperstructures of order 2

- **Classification**

| CYCLIC & ABELIAN | CYCLIC & NON ABELIAN |
|---|---|
| 9 | 10 |
| PROJECTIVE & ABELIAN<br>1 | |

# Hyperstructures of order 3 with scalar unit

- 40353607 possible configurations
- 15322445 candidates verifying weak associativity
- 10323979 candidates verifying axiom of reproduction
- 1677 hyperstructures

**Theorem (2004)**

There are 292 non isomorphic hyperstructures of order 3 with scalar unit.

# Hyperstructures of order 3 with scalar unit

- Classification

<table>
<tr><td colspan="2">CYCLIC & ABELIAN<br>74</td><td colspan="2">CYCLIC & NON ABELIAN<br>205</td></tr>
<tr><td colspan="2">NON CYCLIC & ABELIAN</td><td colspan="2">NON CYCLIC & NON ABELIAN</td></tr>
<tr><td>PROJECTIVE<br>1</td><td>NON PROJECTIVE<br>8</td><td>PROJECTIVE<br>0</td><td>NON PROJECTIVE<br>4</td></tr>
</table>

# Divisibility

**Theorem (2004)**

If H is a hyperstructure with a scalar unit then $\mathbf{Z}_n$ is a subgroup of Aut(H).

# References


- Birkhoff G., 1946. *Sobre los grupos de automorphimos.* Revista Union Math. Arg., 11. pp.155-157.
- Chaunier C. and N. Lygeros 1994. *Posets minimaux ayant un groupe d'automorphismes d'ordre premier.* C.R.Acad.Sci. Paris, t.318, Série I, p. 695-698.
- De Salvo M. and Freni D. 1981. *Semi-ipergruppi ciclici.* Atti Sem. Mat. Fis. Un. Modena, 30, pp.44-59.
- Lygeros N. and Mizony M.1996. *Construction de posets dont le groupe d'automorphismes est isomorphe à un groupe donné.* C.R.Acad.Sci. Paris, t.322, Série I, pp. 203-206.
- Lygeros N.  2004. *Posets, Groups and Hypergroups.* Perfection vol.5 8.
- Lygeros N.  2004. *Sur les hypergroupes rigides.* Perfection vol.5 9.

# References

- Marty F. 1934. *Sur une généralisation de la notion de groupe.* 8th Congress Math. Scandinaves, Stockholm, pp.45-49.
- Mittas J. 1984 *Hypergroups renaissants.* Atti. Sem. Mat. Fis. Un. Modena, 33,319-328.
- Vougiouklis Th. 1990. *Isomorphisms on P-hypergroups and cyclicity.* Ars Combinatoria 29A , pp.241-245.
- Vougiouklis Th. 1981. *Cyclicity in a special class of hypergroups.* Acta Un. Car.-Math. Ph., 22, N.1, 3-6.
- Vougiouklis Th. 1980. *Hypergroups cyclicity* ;.Thesis 1980.
- Vougiouklis Th., Spartalis S., Kessoglides M. 1997: *Weak Hyperstructures on small sets.* Ratio Mathematica 12, 90-96.