The H_v-groups and Marty-Moufang Hypergroups

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Abstract

We partition, enumerate and classify H_v-groups of order 2 (20) and 3 (1.026.462) as well as abelian H_v-groups of order 4 (8.028.299.905), thus generalizing Migliorato and Nordo results on hypergroups, and Choi and Chung results on minimal H_v-groups. Then, after introducing the Marty-Moufang hypergroups that algebraically generalize Marty's hypergroups and Moufang's loops, we enumerate those of order 2 (10) and 3 (96.058). Finally we interpret our results via the posets theory and their circle inclusion representation in specific cases.

1 Introduction and definition

Definition 1 (F. Marty [22, 23, 24]). *An hypergroup $< H, . >$ is a set H equipped with an associative hyperoperation $(.) : H \times H \longrightarrow \mathcal{P}(H)$ which satisfies the reproduction axiom : $xH = Hx = H$ for all x in H.*

Definition 2 (Th. Vougiouklis [30]). *An hyperstructure $< H, . >$ is called an H_v-group if the following axioms hold :*

(i) $x(yz) \cap (xy)z \neq \emptyset$ for all x, y, z in H (weak associativity)

(ii) $xH = Hx = H$ for all x in H (reproduction)

2 Enumeration of H_v-groups

2.1 Order 2

Theorem 1 (R. Bayon-N. Lygeros [2]). *There are 20 isomorphism classes of H_v-groups of order 2 (see table 1).*

H_v-group	$\lvert Aut(H_v)\rvert$	H_v-group	$\lvert Aut(H_v)\rvert$
$(a;b;b;a)^*$	2	$(H;a;H;b)^*$	2
$(H;b;b;a)$	2	$(a;H;H;b)^*$	1
$(a;H;b;a)$	2	$(H;a;a;H)$	2
$(a;b;H;a)$	2	$(H;b;a;H)$	1
$(H;a;a;b)^*$	2	$(H;a;b;H)$	1
$(H;H;b;a)$	2	$(H;H;H;a)^*$	2
$(H;b;H;a)$	2	$(H;H;H;b)^*$	2
$(a;H;H;a)$	2	$(H;H;a;H)$	2
$(b;H;H;a)$	1	$(H;H;b;H)$	2
$(H;H;a;b)^*$	2	$(H;H;H;H)^*$	1

Table 1: H_v-groups of Order 2 ($H = \{a, b\}$)

Compared to Th. Vougiouklis [32] we add the two following H_v-groups : (H, b, a, H) and (b, H, H, a) who are rigids (i.e. their automorphism groups are trivial) [1, 16, 17, 18].

2.2 Order 3

Theorem 2 (R. Bayon-N. Lygeros [2]). *There are 1.026.462 isomorphism classes of H_v-groups of order 3 (see table 2).*

		Classes					
		Abelians			non-Abelians		
		Cyclics	non-Cyclics		Cyclics	non-Cyclics	
			Proj.	non-Proj.		Proj.	non-Proj.
$\|Aut(H_v)\|$	1	5	2	-	4	2	-
	2	8	1	1	47	5	7
	3	243	8	14	2034	66	76
	6	7439	10	195	1003818	1083	11394

Table 2: Classification of H_v-groups of Order 3

2.3 Order 4

Theorem 3 (R. Bayon-N. Lygeros [6]). *There are 10.614.362 isomorphism classes of abelian hypergroups of order 4.*

Theorem 4 (R. Bayon-N. Lygeros [3]). *There are 8.028.299.905 isomorphism classes of abelian H_v-groups of order 4 (see table 3).*

		Classes		
		Cyclics	non-Cyclics	
			Proj.	non-Proj.
$\|Aut(H_v)\|$	1	5	3	-
	2	-	-	-
	3	38	5	6
	4	582	22	39
	6	2.215	45	144
	8	2.149	39	144
	12	1.859.161	1.827	39.773
	24	7.994.020.227	86.159	32.287.322

Table 3: Classification of abelian H_v-groups of Order 4

3 The Marty-Moufang Hypergroups

From the group theory of E. Galois [15], we can generalize the concept of neutral element via the methodology of F. Marty who introduces the axiom of reproduction. We thus notice that in this generalization associativity is not affected. It is also natural to generalize it *a posteriori*. This is the main work of Th. Vougiouklis who created the H_v-groups and thereafter the H_v-structures, by replacing the equality by a nonnull intersection in associativity. However, this way, even if leading to many results, is not necessarily universal. From an algebraic point of view, we introduce a new generalization of hypergroup :

Definition 3 ([19, 20]). *An hyperstructure* $< H, . >$ *is called a Marty-Moufang hypergroup if the reproduction axiom is valid and (.) verifies the Moufang identity [27, 28] :* $(xy)(zx) = x((yz)x)$.
These hyperstructures are noted H_m-groups.

3.1 Order 2

Theorem 5. *There are 10 isomorphism classes of Marty-Moufang hypergroups of order 2 (see table 4).*

H_m-group	$\|Aut(H_m)\|$
$(a; b; b; a)^*$	2
$(H; H; H; a)^*$	2
$(H; a; a; b)^*$	2
$(H; H; a; b)^*$	2
$(H; a; H; b)^*$	2
$(a; H; H; b)^*$	1
$(H; H; H; b)^*$	2
$(H; H; a; H)$	2
$(H; H; b; H)$	1
$(H; H; H; H)^*$	1

Table 4: Isomorphism classes of Marty-Moufang Hypergroups of order 2.

3.2 Order 3

Theorem 6. *There are 96.058 isomorphism classes of* H_m*-groups of order 3 (see table 5).*

$\|Aut(H_m)\|$	1	2	3	6
	10	30	770	95.248

Table 5: Number of Marty-Moufang Hypergroups isomorphism classes relatively to the order of their automorphism groups

Remark 1. *(H,bc,ac,ac,bc,ab,bc,a,a) is a* H_m*-group but it is not a* H_v*-group :* $c(bb) = \{a\}$ *and* $(cb)b = \{b, c\}$.

4 Posets defined on hyperstructures

C. Chaunier-N. Lygeros, as well as R. Fraïssé-N. Lygeros, have counted posets on up to 14 elements [10, 11, 12, 14]. Thanks to this work R. Bayon-N. Lygeros-J.S. Sereni enumerated mixed models [7, 8]. R. Fraïssé and N. Lygeros studied circle order [14], and R. Bayon-N. Lygeros-J.S. Sereni proved that all orders on at most 10 elements are circle orders [7]. N. Lygeros-M. Mizony studied posets having a given automorphism group [9, 21].

Definition 4 (Th. Vougiouklis [31]). *An hyperoperation* $(.)$ *is called smaller than the hyperoperation* $(*)$*, and written as* $. < *$*, if and only if there is an* $f \in Aut(H, *)$ *such that* $xy \subseteq f(x * y)$ *for all* x, y *in* H*.*
He defines too the notion of minimality [33] : An hyperoperation is called minimal if it contains no other hyperoperation defined on the same set. So we can construct posets defined on set of hyperstructures.

Theorem 7 (Th. Vougiouklis [31]). *A greater hyperoperation than the one of a given* H_v*-group defines an* H_v*-group.*

Definition 5 (J. Mittas [26]). *An hypergroup is called canonical if following axioms hold :*

(i) $x(yz) = (xy)z$ *(associativity)*

(ii) $xy = yx$ *(commutativity)*

(iii) There exists $1 \in H$ *such that, for all* $x \in H$*,* $x.1 = x$ *(scalar unit)*

(iv) For all $x \in H$*, there exists one and only one* $x' \in H$ *such that,* $1 \in xx'$ *(*x' *will be noted* x^{-1} *and* $x/y = xy^{-1}$*) (inverse)*

(v) $z \in xy \Rightarrow y \in z/x$ *(reciprocity).*

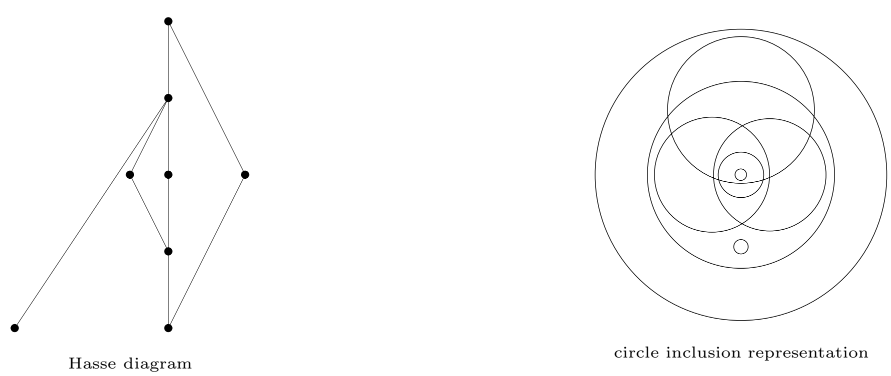

Figure 1: Poset of Hypergroups of Order 2

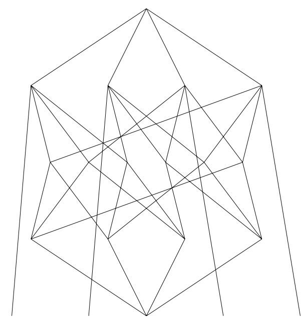

Figure 2: Poset of H_v-groups of Order 2

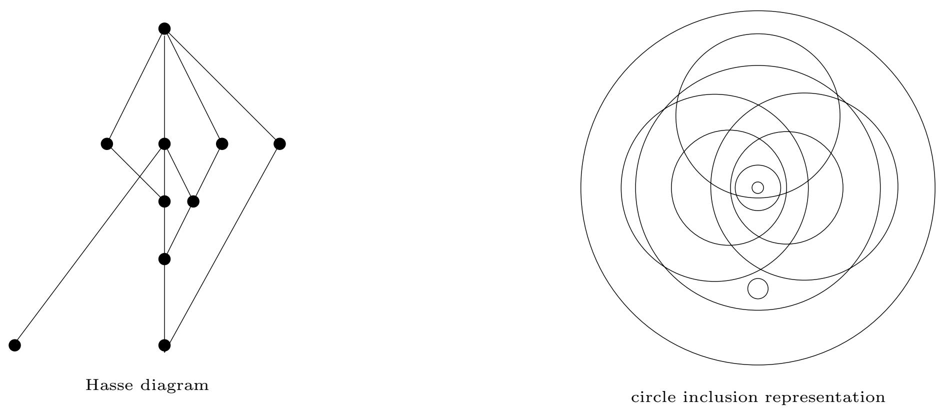

Figure 3: Poset of H_m-groups of Order 2

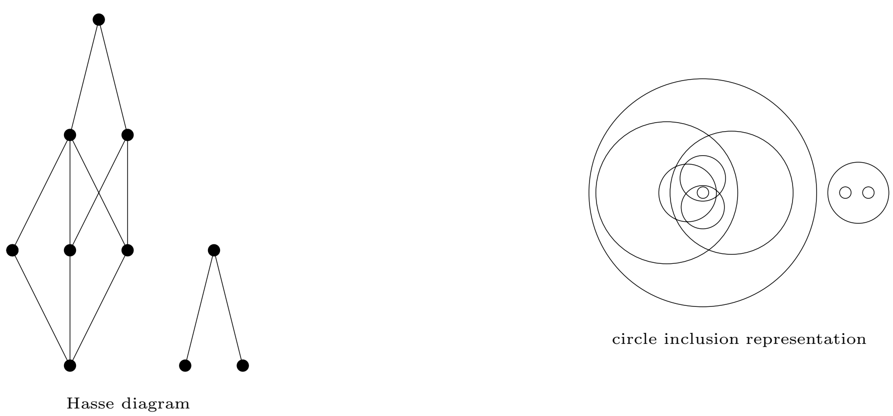

Figure 4: Poset of Canonical Hypergroups of Order 3

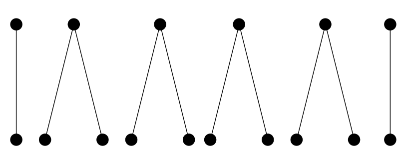

Figure 5: Poset of Very Thin H_v-groups of Order 3

5 Algorithm

We developed two different algorithms for the enumeration of hyperstructures. The first one constructs posets of hyperstructures and is similar to the algorithm of G. Nordo [29]. The second one, based on the explicit computation of the automorphism group, allows us to enumerate hyperstructures at order 4, as shown in our precedent work [6].

5.1 Poset construction

An hyperstructure is represented by a list of integers. Each integer represents an element of the Cayley table of an hyperoperation. We generate all the possible hyperoperations. During this step the reproduction axiom is checked dynamically. If the reproduction axiom is valid we check the Moufang identity (or associativity or weak associativity). When both properties are valid, we compute the partition of the hyperoperations. That means we partition the hyperoperations relatively to the number of hyperproducts of a given order. This partitioning reduces the number of isomorphism tests. It speeds up too the posets construction.

5.2 Enumeration

The previous algorithm is necessary to construct the poset of hyperstructures. Indeed, we need to know all the isomorphisms of hyperstructures. But to obtain enumerative results we developed a new algorithm.

We compute the whole set of hyperstructures and for each hyperstructure we determine its automorphism group. Using the following formula, we obtain the number of hyperstructures :

$$p = \sum_{i=1}^{n!} \frac{s_i}{i}$$

Where n is the order of hyperstructures, s_i is the number of hyperstructures with automorphism group of order i.

5.3 Validation

With our algorithms we get the result of R. Migliorato [25], who computes the 23192 hypergroups of order 3 and the result of G. Nordo who obtains, up to isomorphism, the 3999 hypergroups of order 3 [4, 5]. As S-C. Chung and B-M. Choi [13] we get, at order 3, 13 minimal H_v-groups with scalar unit.

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