On the compact form of Catalan Numbers
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Catalan numbers turn up in many types of problems as the Euler's polygon division triangles or the solution to the ballot problem. But originally the Catalan numbers solve Catalan problem i.e.enumeration of binary bracketings. As they are isomorphic to binary trees with their enumeration we have the following recursive system: $a_1 = a_0.a_0 a_2 = a_0 a_1 + a_1 a_0$ and $a_n = \sum\limits_{0 \le k \le n-1} a_k a_{n-1-k} \quad n \ge 2$

So the generating function of this integer sequence is:

$$C(t) = \sum_{n=0}^{\infty} a_n t^n = 1 + t + \sum_{n=2}^{\infty} a_n t^n = 1 + t + \sum_{n=2}^{\infty} t^n \left(\sum_{0 \le k \le n-2} a_k a_{n-1-k} \right)$$

$$C(t) = 1 + t \sum_{n=1}^{\infty} t^{n-1} \left(\sum a_k a_{k+1} \right)$$

$$C(t) = 1 + (-1)t \sum a_k . a_k . t^{k+h} = 1 + t - 2t + \left(\sum_0 a_k t^k \right) \left(\sum_0 a_h t^h \right)$$

Let $n=k+h$, we get: $C(t) = - t + C^2(t)$

$\Delta = 1+4t$ and as $C(0) = 1$ we get $: C(t) = (+1 - O\,(1+4t)\,)\,/\,2$

$$\left(1+t\right)^{\alpha} = 1 + \sum_{n \ge 1} \frac{\alpha(\alpha-1)(\alpha-2)....(\alpha-n+1)}{n!} t^n = 1 + \sum_{n=0}^{\infty} C_{\alpha}^{n} t^n = \sum_{n=0}^{\infty} C_{\alpha}^{n} t^n$$

$$\left(1+t\right)^{1/2} = 1 + \sum_{n \ge 1} \frac{1.3....(2n-3)}{2^n n!} t^n$$

So the compact form of **Catalan numbers** is: $C_n = \dfrac{1}{n+1} C_{2n}^{n}$