Κατανομές Maxwell-Boltzmann, Bose-Einstein και Fermi-Dirac

Ν. Λυγερός

Κατανομή Maxwell-Boltzmann

$$\Omega_{MB} = N!\prod_i \frac{g_i^{n_i}}{n_i!}$$

$$\ln\Omega_{MB} = \ln N!\prod_i \frac{g_i^{n_i}}{n_i!}$$

$$\ln\Omega_{MB} = \ln N! + \sum_i \ln g_i^{n_i} - \sum_i \ln n_i!$$

$$\ln\Omega_{MB} = \ln N! + \sum_i n_i \ln g_i - \sum_i n_i \ln n_i + \sum_i n_i$$

$$\ln\Omega_{MB} = \ln N! + \sum_i n_i \ln g_i - \sum_i n_i \ln n_i + N$$

$$Max\Omega_{MB} \Leftrightarrow Max\ln\Omega_{MB} \Leftrightarrow \frac{d\Omega}{dn_i} = 0$$

Περιορισμοί: $\sum n_i = N, \sum n_i E_i = E$

Ιδέα: $Max\Omega_{MB} \Leftrightarrow Max\ln\Omega_{MB} \Leftrightarrow \frac{d\Omega_{MB}}{dn_i} = 0$

Μέσω των πολλαπλασιαστών του Lagrange έχουμε:

$$\frac{d}{dn_j}\left[\ln\Omega_{MB} + \alpha\sum n_i + \beta\sum n_i E_i\right] = 0$$

$$\frac{d}{dn_j}\left(\sum_i n_i \ln g_i\right) - \frac{d}{dn_j}\left(\sum n_i \ln g_i\right) - \frac{d}{dn_j}\left(\sum_i n_i\right) + \frac{d}{dn_j}\left(\alpha\sum_i n_i\right) + \frac{d}{dn_j}\left(\sum_i n_i E_i\right)$$

$$\ln g_i - \ln n_i - \alpha - \beta E_i = 0$$

$$n_i = g_i e^{-\alpha}.e^{-\beta E_i}$$

Όπως:

$$N = \sum_i n_i = \sum_i g_i e^{-\alpha} e^{-\beta E_i}$$

$$N = e^{-\alpha}\sum_i g_i e^{-\beta E_i}$$

$$e^{\alpha} = \frac{N}{\sum_i g_i e^{-\beta E_i}}$$

Κατά συνέπεια: $n_i = \frac{Ng_i}{z}.e^{-\beta E_i}$

Όπου $z = \sum_i g_i e^{-\beta E_i}$

Κατανομή Bose-Einstein

$$\Omega_{BE} = \prod_i \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!}$$

$$\ln\Omega = \ln\prod_i \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} = \sum_i \ln(n_i + g_i - 1)! - \sum_i \ln(n_i!(g_i - 1)!)$$

$$\ln\Omega = \sum_i \left\{(n_i + g_i - 1)\ln(n_i + g_i - 1) - (n_i + g_i - 1)\right\}$$

$$-\sum_i \left[n_i + \ln(n_i) - n_i\right] - \sum_i \left[(g_i - 1)\ln(g_i - 1)(g_i - 1)\right]$$

$$\frac{d\Omega}{dn_i} = 0 = \ln(n_i + g_i - 1) - \ln n_i - \alpha - \beta E_i = 0$$

$$\ln\left(\frac{n_i - g_i - 1}{n_i}\right) = \alpha + \beta E_i$$

Λόγω $n_i, g_i \gg 1$ $\qquad \ln\left(\frac{n_i + g_i}{n_i}\right) = \alpha + \beta E_i$

$$n_i = \frac{g_i}{-1 + e^{\beta(E_i - \mu)}}$$

Κατανομή Fermi-Dirac

$$\Omega = \prod_{i=1}^{N_L} \frac{g_i!}{(g_i - n_i)!n_i!}$$

$$N=\sum_{i=1}^{N_L} n_i \quad E_T=\sum_{i=1}^{N_L} n_i E_i$$

$$S=k\ln(\Omega)=k\ln\left(\prod_{i=1}^{N_L}\frac{g_i\,!}{(g_i-n_i)!n_i\,!}\right)=k\sum_{i=1}^{N_L}\left[\ln(g_i\,!)-\ln(n_i\,!)-\ln(g_i-n_i)!\right]$$

Προσέγγιση του Stirling: $\ln(n!)=n\ln(n)-n$

$$S=k\ln(\Omega)=k\sum_{i=1}^{N_L}\left\{[\ln(g_i)!]-n_i\ln(n_i)+\cancel{n_i}-(g_i-n_i)\ln(g_i-n_i)+g_i-\cancel{n_i}\right\}$$

$$S=k\ln(\Omega)=k\sum_{i=1}^{N_L}\left\{[\ln(g_i)!]-n_i\ln(n_i)-(g_i-n_i)\ln(g_i-n_i)+g_i\right\}$$

g_i:σταθερές και $(\max S)\Leftrightarrow(dS=0)$

$$0=dS=k\sum_i\left\{-\ln(\overline{n}_i)-\cancel{1}+\ln(g_i-\overline{n}_i)+\cancel{1}\right\}\delta n_i$$

$$0=k\sum\ln\left(\frac{g_i-\overline{n}_i}{\overline{n}_i}\right)\delta n_i$$

Όπως

$$\sum_i\delta n_i=0 \quad \kappa\alpha\iota \quad \sum_i E_i\delta n_i=0$$

$$a\sum_i\delta n_i=0 \quad \kappa\alpha\iota \quad \beta\sum_i E_i\delta n_i=0$$

Κατά συνέπεια:

$$\sum\left\{\ln\left(\frac{g_i-\overline{n}_i}{\overline{n}_i}\right)+a+\beta E_i\right\}\delta n_i=0$$

Μέσω Lagrange έχουμε:

$$\ln\left(\frac{g_i-\overline{n}_i}{\overline{n}_i}\right)+a+\beta E_i=0$$

Άρα:

$$\frac{g_i-\overline{n}_i}{\overline{n}_i}=e^{a+\beta E_i}$$

$$g_i=\overline{n}_i\left(1+e^{a+\beta E_i}\right)\Leftrightarrow\overline{n}_i=\frac{g_i}{1+e^{a+\beta E_i}}$$