4231) Περί διαχωρισμού λύσεων της εξίσωσης (N. Λυγερός)

□ $\psi = 0$.

Sur la séparation des solutions de l'équation

□ $\psi = 0$.

Έστω: $\psi(x_1, x_2, \dots, x_n, t)$

$$\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_n^2} = \frac{\partial^2 \psi}{\partial t^2}$$

Εάν $\psi = \psi(r,t)$ με $r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$

Έχουμε $\frac{\partial r}{\partial x_j} = \frac{x_j}{r}$

και $\frac{\partial^2 r}{\partial x_j^2} = \frac{r^2 - x_j^2}{r^2}$

$$\sum_{j=1}^{n} \left(\frac{\partial r}{\partial x_j}\right)^2 = 1 \text{ και } \sum_{j=1}^{n} \frac{\partial^2 r}{\partial x_j^2} = \frac{n-1}{r}$$

Έτσι: $\frac{\partial \psi}{\partial x_j} - \frac{\partial \psi}{\partial r} \cdot \frac{\partial r}{\partial x_j}$ και $\frac{\partial^2 \psi}{\partial x_j^2} = \frac{\partial \psi}{\partial r} \cdot \frac{\partial^2 r}{\partial x_j^2} + \frac{\partial r}{\partial_{x_i}} \cdot \frac{\partial \psi}{\partial x_j \cdot \partial r}$

$$\frac{\partial^2 \psi}{\partial r \cdot \partial_{x_i}} = \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial x_j}\right) = \frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial r} \cdot \frac{\partial r}{\partial x_j}\right) = \frac{\partial \psi}{\partial r} \cdot \frac{\partial^2 r}{\partial r \cdot \partial x_j} + \frac{\partial r}{\partial x_j} \cdot \frac{\partial^2 \psi}{\partial r^2}$$

$$\frac{\partial^2 r}{\partial r \cdot \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{\partial r}{\partial r}\right) = 0 \Rightarrow \frac{\partial^2 \psi}{\partial r \cdot \partial x_j} = \frac{\partial r}{\partial x_j} \cdot \frac{\partial^2 \psi}{\partial r^2}$$

Κατά συνέπεια: $\frac{\partial^2 \psi}{\partial x_j} = \frac{\partial \psi}{\partial r} \frac{\partial^2 r}{\partial x_j^2} + \left(\frac{\partial r}{\partial x_j}\right)^2 \frac{\partial^2 \psi}{\partial r^2}$

$$\sum_{j=1}^{n} \frac{\partial^2 \psi}{\partial_{x_i^2}} = \frac{\partial \psi}{\partial r} \sum_{j=1}^{n} \frac{\partial^2 r}{\partial x_j^2} + \frac{\partial^2 \psi}{\partial^2 r} \sum_{j=1}^{n} \left(\frac{\partial r}{\partial x_j}\right) = \left(\frac{n-1}{r}\right) \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2}$$

Άρα: $\frac{\partial^2 \psi}{\partial r^2} + \left(\frac{n-1}{r}\right) \frac{\partial \psi}{\partial r} = \frac{\partial^2 \psi}{\partial t^2}$

Έστω: $\varphi(r,t) = r^k \psi(r,t)$

$\frac{\partial \varphi}{\partial r} = r^k \frac{\partial \psi}{\partial r} + k r^{k-1} \psi$ και $\frac{\partial^2 \varphi}{\partial r^2} = r^k \frac{\partial^2 \psi}{\partial r^2} + 2k e^{k-1} \frac{\partial \psi}{\partial r} + k(k-1) r^{K-2}\, \psi$

Έστω k=$\frac{n-1}{2}$. Έχουμε μετά από διαίρεση με το r^k:

$$\frac{1}{r^{\frac{n-1}{2}}} \cdot \frac{\partial \varphi^2}{\partial r^2} = \frac{\partial^2 \psi}{\partial r^2} + \frac{n-1}{r}\frac{\partial \psi}{\partial r} + \frac{(n-1)(n-3)}{4r^2} = \frac{\partial^2 \psi}{\partial t^2}$$

Άρα: $\frac{\partial \varphi^2}{\partial r^2} - \frac{(n-1)(n-3)}{4r^2}\varphi = \frac{\vartheta^2 \varphi}{\partial t^2}$

Εάν:
- $n = 1$: Έχουμε: $\psi(r,t) = f(r-t) + g(r+t) \quad \forall f, g$
- $n = 3$: Έχουμε: $\psi(r,t) = \frac{f(r-t)}{r} + g\frac{(r+t)}{r} \quad \forall f, g$