Euler-Lagrange & Maxwell

Nikos Lygeros

$\square A_\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu_{em}$ (Euler-Lagrange)

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\mathcal{L}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\nu_{em} A_\nu$ (Lagrange)

$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ (Maxwell - Lagrange)

$(j_{em} = 0) \Longrightarrow (\partial_\mu F^{\mu\nu} = \square A^\nu - \partial^\nu(\partial^\mu A_\mu) = 0)$

Gauge: $A^\mu \longrightarrow A'^\mu = A^\mu - \partial^\mu \chi \Longrightarrow \partial_\mu A^\mu = 0$ (Lorentz)

$(\square A^\mu = 0) \Longrightarrow A^\mu = N_\varepsilon{}^\mu e^{-ikx}$ $\quad k^2 = 0$

$(\partial_\mu A^\mu = 0) \Longrightarrow k \cdot \varepsilon = 0$

$A^\mu \longrightarrow A^\mu - \partial^\mu \tilde{\chi}$ $\quad \square\tilde{\chi} = 0$

$\varepsilon^\mu \longrightarrow \varepsilon^\mu + \beta k^\mu \equiv \varepsilon'^\mu$ $\quad (k^2 = 0) \Longrightarrow \varepsilon'^\mu \cdot k = 0$

Classique

$$A^\mu(x) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega}}\left[\varepsilon^\mu(\kappa,\lambda)\alpha(\kappa,\lambda)e^{-ikx} + \varepsilon^{\mu *}(\kappa,\lambda)\alpha^*(\kappa,\lambda)e^{-ikx}\right]$$

Quantique

$$\hat{A}^\mu(x) = \sum_{\lambda=0}^{3} \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega}}\left[\varepsilon^\mu(\kappa,\lambda)\hat{\alpha}_\lambda(\kappa)e^{-ikx} + \varepsilon^{*\mu}(\kappa,\lambda)\hat{\alpha}^\dagger_\lambda(\kappa)e^{-ikx}\right]$$

Gauge: $\partial^\mu \longrightarrow \hat{D}^\mu = \partial^\mu + iq\hat{A}^\mu$

(Dirac - Lagrange) $\hat{\mathcal{L}}_D = \overline{\hat{\psi}}(i\gamma^\mu\partial_\mu - m)\hat{\psi}$

U(1) $\hat{\psi}(x,t) \longrightarrow \hat{\psi}'(x,t) = e^{-iq\hat{\chi}(x,t)}\hat{\psi}(x,t)$

$\hat{A}^\mu \longrightarrow \hat{A}'^\mu = \hat{A}^\mu + \partial^\mu\tilde{\chi}$

U(1) – invariant $\hat{\mathcal{L}}_{D\ local} = \overline{\hat{\psi}}(i\gamma^\mu\hat{D}_\mu - m)\hat{\psi}$

$\hat{D}'_\mu\hat{\psi}' = e^{-iq\hat{\chi}}(\hat{D}_\mu\hat{\psi})$

$(i\gamma^\mu\hat{D}_\mu - m)\hat{\psi}' = e^{-iq\hat{\chi}}(i\gamma^\mu\hat{D}_\mu - m)\hat{\psi}$

$$\hat{\bar{\psi}}' = \hat{\bar{\psi}} e^{iq\hat{\chi}}$$

$$\hat{\bar{\psi}}' \left(i\gamma^{\mu} \hat{D}'_{\mu} - m\right)\hat{\psi}' = \hat{\bar{\psi}} e^{iq\hat{\chi}} e^{-iq\hat{\chi}} \left(i\gamma^{\mu} \hat{D}_{\mu} - m\right)\hat{\psi} = \hat{\bar{\psi}}\left(i\gamma^{\mu} \hat{D}_{\mu} - m\right)\hat{\psi}$$

$$\hat{\mathcal{L}}_D \longrightarrow \hat{\mathcal{L}}_{D\ local} = \hat{\mathcal{L}}_D + \hat{\mathcal{L}}_{int} \quad \hat{\mathcal{L}}_{int} = -q\hat{\bar{\psi}}\gamma^{\mu}\hat{\psi}\ \hat{A}_{\mu}$$

(Hamilton) $\hat{H} = \hat{H}_D + \hat{H}'_D$

$$\hat{H}'_D = -\hat{\mathcal{L}}_{int} = q\hat{\bar{\psi}}\gamma^{\mu}\hat{\psi}\ \hat{A}_{\mu} = q\hat{\psi}^{+}\hat{A} - q\hat{\psi}^{+}\alpha\hat{\psi}\cdot\hat{A}$$

$$\hat{j}^{\mu}_{em} = q\hat{\bar{\psi}}\gamma^{\mu}\hat{\psi} = q\hat{N}^{\mu}_{\psi}$$