Posets, Groups and Hypergroups

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Abstract - Via the transposition of the theorems of Birkhoff, Chaunier-Lygeros and Lygeros-Mizony on Groups to P-Hypergroups, we explicit some relations between posets and hypergroups.

1. Definitions
is a hypergroup if : $H * H \to p(H)$ is an associative hyperoperation for which the reproduction axiom $hH = Hh = H$ is valid for every h of H [5].

A hypergroup $(H, .)$ is called **cyclic** with finite period with respect an element h of H if there exists an integer v such that $H = h^1 \cup h^2 \cup ... \cup h^v$[3].

Let $(G, .)$ be a group and P any non-empty subset of G. Then the P-hyperoperation $P*$ is defined as follows : $P* : G * G \to p(G) : (x, y) \to xP * y = xPy$. If the reproduction axiom is valid then the hyperstructure is a P-**hypergroup** [6].

2. Theorems
Theorem Birkhoff [1] : If G is group then there exists a poset which automorphisms group is isomorphic to G.

Theorem Birkhoff [1] : If G is a finite group of cardinal a then there exists a poset which automorphisms group is isomorphic to G and with cardinal is equal to $a^2 + a$.

Theorem Chaunier-Lygeros [2] : Let **a** a prime number and a the minimal number of vertices of posets which have a automorphisms group of cardinality **a**. Then :
(i) $n = a$ if $a = 2$;
(ii) $n = 3a$ if $a = 3, 5$ or 7;
(iii) $n = 2a$ if $a > 7$.

Posets which realize the minimum and have a minimal number of relations are respectively :
(i) $(\{x_0, x_1\})$ with x_0 and x_1 incomparable;
(ii) $(\{x_0, \ldots, x_{a-1}, y_0, \ldots y_{a-1}, z_0, \ldots, z_{a-1}\})$ with $x_i < y_i < z_i$ and $x_i < z_j$ if $j - i = 1 \pmod a$;
(iii) $(\{x_0, \ldots, x_{a-1}, y_0, \ldots y_{a-1}\})$ with $x_i < y_j$ if $j - i = 0, 1$ or $3 \pmod a$.

Theorem Lygeros-Mizony [5] : If G is a finite group of cardinal a, non direct product of two groups, generated by elements which are two by two of distinct order then there exists a poset which automorphisms group is isomorphic to G and with cardinal is equal to $3a$.

3. New Theorems
Theorem 3.1 : A poset Po can be associated to every P-hypergroup. Proof : By the definition of a P-hypergroup we have : $G \leftrightarrow P$-hypergroup and with the theorem of Birkhoff [1] we can associate a poset Po to a group G via its automorphisms group $Aut(Po)$. So we have : $Po \leftrightarrow Aut(Po) \sim G$
Q.E.D.

Theorem 3.2 : A poset Po of cardinality $a^2 + a$ can be associated to every P-hypergroup with G of cardinality a.
Proof : Corollary of the second theorem of Birkhoff.
Q.E.D.

Theorem 3.3 : Let a a prime number, P-hypergroup and G a group of cardinality $a = 2$ or 3,5,7 or 11 and more then there is an associated poset Po of cardinality respectively a or $3a$ or $2a$ which has an automorphism group of order a.
Proof : Corollary of the theorem of Chaunier-Lygeros.
Q.E.D.

Theorem 3.4 : Let P-hypergroup and G a finite group of cardinal a, non direct product of two groups, generated by elements which are two by two of distinct order then there exists a poset which automorphisms group is isomorphic to G and with cardinal is equal to $3a$.
Proof : Corollary of the theorem of Lygeros-Mizony.
Q.E.D.

4. Questions

Question 1 : Can we associate a general hypergroup to a poset ?
Question 2 : Can we associate a poset to a general hypergroup ?

References

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