# Advanced Results in Enumeration of Hyperstructures

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**Abstract**

We first recall our results on enumeration of hypergroups and $H_v$-groups of order 2, 3 and 4. Then we carry out a study on posets of hypergroups and $H_v$-groups. These results are extended to hyperrings. We finally explain the algorithms used.

*Key words:* abelian, classification, enumeration, automorphisms group, hypergroup, $H_v$-group, hyperring ,poset

## 1  Introduction and Definitions

More than seventy years have gone by since the creation of the concept of hypergroup by F. Marty in 1934 [21]. However the origin of this creation remains still unclear due to the mysterious veil which covers a part of the life of its author. The implicit claims on behalf of M. Krasner and of H.S. Wall, contribute to maintain some blurring. But their presences prove, even if merely in a hazy way, the importance of the discovery of the concept of hypergroup, without sufficing to dispossess F. Marty of the paternity of his creation.

**Definition 1 (F. Marty [21, 22, 23]).** $< H, . >$ *is an hypergroup if* $(.)$ : $H \times H \to p(H)$ *is an associative hyperoperation for which the reproduction axiom* $hH = Hh = H$ *is valid for any h of H.*

The concept developed by H.S. Wall in 1937 [41] is rather close. We compare the two concepts so as to identify the crosschecking points and the diverging points. The concept of hypergroup of H. S. Wall is based on the following four axioms:

(i) The product postulate: The product of two elements of $H$ is a complex (in the sense of an assembly) of $n$ elements of $H$ uniquely determined.

(ii) The associative law: If $(a, b, c)$ are three elements of $H$ then $a(bc) = (ab)c = abc$.

(iii) The identity postulate: There is at least an element $e$ in $H$ such as for any element $a$ of $H$ the $ae$ and $ea$ products contain at least both the element $a$.

(iv) The postulate of the inverse: There is at least an element $a^{-1}$ in $H$ such as for any element $a$ of $H$ the products $aa^{-1}$ and $a^{-1}a$ contain at least both the element $e$.

Whereas the concept of hypergroup of F. Marty is based only on two axioms, i.e. axiom of reproduction and associative law. Thus the associative law is present in the two concepts. On the other hand the identity postulate and the inverse postulate, which are interpretable in more modern terms as weak properties since there is not necessarily equality but nonempty intersection, are completely absorbed in F. Marty's hypergroup structure since it has a meaning even in the absence of the neutral element and at the same time of the opposite element. So there only remains the postulate of the product to differenciate the two hypergroup concepts. Indeed the restriction on the three other axioms of H. S. Wall clearly shows that the generated class is included in the class generated by the axioms of F. Marty. The postulate of the product makes it possible to introduce the concept of multiplicity and it is this one which characterizes the formalism chosen by H. S. Wall. Nevertheless, if the elements all are different then the H. S. Wall's class is included again in the F. Marty's class. This body of comparisons, even if it cannot be completely ordered, shows the richness of F. Marty's definition to create the concept of hypergroup [19]. In 1991 Th. Vougiouklis generalized the definition of F. Marty by weakening associativity [33].

**Definition 2.** *An hyperoperation is weakly associative if for any $x, y, z \in H$, $x(yz) \cap (xy)z \neq \emptyset$.*

**Definition 3 (Th. Vougiouklis [33]).** $< H, . >$ *is a $H_v$-group if $(.) : H \times H \to p(H)$ is a weakly associative hyperoperation for which the reproduction axiom $hH = Hh = H$ is valid for any $h$ of $H$.*

The essential idea which governs the existence of these $H_v$-groups is a weakening associativity. This weakening simply consists in considering the two terms of the associative law as sets, since this is possible, and in requiring that their intersection shall not be empty. Those objects have been studied by Th. Vougiouklis [14, 34, 37, 38, 39]. This idea was extended to more general structures, like the hyperrings and the hyperfields [17, 29, 30].

The $H_v$-groups have a property of which the hypergroups are deprived. This one is built from the definition of the following partial order.

**Definition 4 (Th. Vougiouklis [34]).** *Let $< H, . >$ and $< H, * >$ two $H_v$-groups. We say that $(.)$ is less or equal than $(*)$, and note $\leq$, if and only if there exists $f \in Aut(H, *)$ such that $xy \subseteq f(x * y)$ for any $x, y$ of $H$.*

From this definition we can deduce the following theorem:

**Theorem 1 (Th. Vougiouklis [34]).** *If an hyperoperation is weakly associative, then any hyperoperation superior to it and defined on the same set is weakly associative too.*

From this property, we can show the concept of minimality in a natural way. The minimal $H_v$-group is the one that verifies this property but which does not contain another defined $H_v$-group on the same set. It is in this manner that have been found the thirteen minimal $H_v$-groups of order 3 containing a neutral element, as we will specify it thereafter. In spite of these results it is obvious that in the field of the enumeration of the hypergroups as well as of the $H_v$-groups, the exploitation of techniques coming from the enumeration of much simpler structures, like the partially ordered sets [12, 20] will allow considerable progress [5, 7, 31]. Indeed, this new approach which concentrates amongst other things on the automorphisms group contains elements able to transcend some combinatoric difficulties.

After our works on the hypergroups enumeration [2, 5, 6, 7], we thus concentrate on the enumeration of the $H_v$-groups of order 2 and 3 as well as that of the abelian $H_v$-groups of order 4.

## 2   The $H_v$-groups of Order 2

All the Th. Vougiouklis contribution concentrates in weak associativity. However the latter does not respect any more the equality that is preserved even in such objects as the quaternions, which are regarded as exotic by some. This replacement of the equality by the nonempty intersection represents a true innovation because it is a breaking point with the traditional approach that is however also based on the set theory. Indeed, to study the heart of a primarily algebraic structure we return to a typically ensemblist idea. Moreover, this new approach enables us to manage a new property which characterizes the $H_v$-groups entity: their heredity compared to the addition or deletion of a new element. This idea, which may seem elementary at first sight and which poses no difficulty in its demonstration, is the base of all thinking about $H_v$-groups. This property allows to classify them in a natural way and to put forward even deeper combinatoric structures. It can also be partially exploited in the field of hypergroups. But without any doubt, its most innovating character is in its creativity, that highlights the universal and complete character of this approach. Thus the $H_v$-groups owe their power and their general information with the characteristic and the weakness of the associativity which plays a central part in this second generalization of the groups. The Th. Vougiouklis idea on weak associativity, although in line with F. Marty's idea on reproduction, remains an innovation because it is *in fine* unforeseeable in its consequences. We examine now the concrete structure of these new entities.

**Theorem 2.** *There exists, up to isomorphism, 20 $H_v$-groups of order 2 (see table 1).*

| $H_v$-group | $|Aut(H_v)|$ | $H_v$-group | $|Aut(H_v)|$ |
|:---:|:---:|:---:|:---:|
| $(a; b; b; a)^*$ | 2 | $(H; a; H; b)^*$ | 2 |
| $(H; b; b; a)$ | 2 | $(a; H; H; b)^*$ | 1 |
| $(a; H; b; a)$ | 2 | $(H; a; a; H)$ | 2 |
| $(a; b; H; a)$ | 2 | $(H; b; a; H)$ | 1 |
| $(H; a; a; b)^*$ | 2 | $(H; a; b; H)$ | 1 |
| $(H; H; b; a)$ | 2 | $(H; H; H; a)^*$ | 2 |
| $(H; b; H; a)$ | 2 | $(H; H; H; b)^*$ | 2 |
| $(a; H; H; a)$ | 2 | $(H; H; a; H)$ | 2 |
| $(b; H; H; a)$ | 1 | $(H; H; b; H)$ | 2 |
| $(H; H; a; b)^*$ | 2 | $(H; H; H; H)^*$ | 1 |

Table 1
List of the $H_v$-groups of order 2 ($H = \{a, b\}$)

Compared to Th. Vougiouklis [35] we have added the following $H_v$-groups: $(H, b, a, H)$ and $(b, H, H, a)$ which are rigid (i.e. their automorphisms group is of order 1).

## 3   The $H_v$-groups of Order 3

**Theorem 3 (S-C. Chung, B-M. Choi [13]).** *There exists, up to isomorphism, 13 minimal $H_v$-groups of order 3 with scalar unit (see table 2).*
We give below the list of these $H_v$-groups in reduced form. That is to say, we consider $< H = \{e, a, b\}, . >$ with scalar unit $e$ and we give the hyperproducts $(aa, ab, ba, bb)$.

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| |
|---|
| $(b, e, e, a)$ |
| $(eb, a, a, e)$ |
| $(e, ab, ab, e)$ |
| $(a, eb, eb, a)$ |
| $(ab, ea, ea, e)$ |
| $(H, eb, a, ea)$ |
| $(H, a, eb, ea)$ |
| $(a, H, H, e)$ |
| $(b, H, H, e)$ |
| $(a, H, H, b)$ |
| $(H, b, a, H)$ |
| $(H, a, b, H)$ |
| $(H, e, ab, H)$ |

Table 2

List of the minimal $H_v$-groups of order 3 with scalar unit ($H = \{e, a, b\}$)

Compared to Th. Vougiouklis, S. Spartalis, and M. Kessoglides [40] we have added the three following $H_v$-groups: $(H, e, ab, H)$, $(H, a, b, H)$, $(H, b, a, H)$. We prove below their minimality.

*Proof.* Let us show the minimality of the $H_v$-group given in reduced form: $(H, ab, e, H)$. Suppose that $(H, ab, e, ea)$ is a minimal $H_v$-group. It verifies the reproduction axiom, but does not verify weak associativity: $b.(a.b) = b.\{b, a\} = b.b \cup b.a = \{e, a\} \cup e = \{e, a\}$ and $(b.a).b = e.b = \{b\}$, contradiction.

The demonstration is similar in all other cases. $\square$

S-C. Chung and B-M. Choi have previously discovered these three $H_v$-groups with a different method.

**Theorem 4.** *There exists, up to isomorphism, 292 $H_v$-groups of order 3 with scalar unit.*

**Theorem 5.** *There exists, up to isomorphism, 6494 minimal $H_v$-groups of order 3 (see table 3).*

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| | | Classes | | | | | |
|---|---|---|---|---|---|---|---|
| | | Abelians | | | non Abelians | | |
| | | Cyclics | non Cyclics | | Cyclics | non Cyclics | |
| | | | Proj. | non Proj. | | Proj. | non Proj. |
| $\|Aut(H_v)\|$ | 1 | 1 | 1 | - | 2 | 2 | - |
| | 2 | 2 | 1 | - | 8 | 2 | 1 |
| | 3 | 11 | 2 | 3 | 90 | 8 | 12 |
| | 6 | 102 | 1 | 13 | 5936 | 47 | 249 |

Table 3
Classification of the minimal $H_v$-groups of order 3

**Theorem 6.** *There exists, up to isomorphism, 1.026.462 $H_v$-groups of order 3 (see table 4).*

| | | Classes | | | | | |
|---|---|---|---|---|---|---|---|
| | | Abelians | | | non Abelians | | |
| | | Cyclics | non Cyclics | | Cyclics | non Cyclics | |
| | | | Proj. | non Proj. | | Proj. | non Proj. |
| $\|Aut(H_v)\|$ | 1 | 5 | 2 | - | 4 | 2 | - |
| | 2 | 8 | 1 | 1 | 47 | 5 | 7 |
| | 3 | 243 | 8 | 14 | 2034 | 66 | 76 |
| | 6 | 7439 | 10 | 195 | 1003818 | 1083 | 11394 |

Table 4
Classification of the $H_v$-groups of order 3

Within the framework of the generalization of the hypergroups in the sense of F. Marty, Th. Vougiouklis introduced the concept of hyperstructure which he named $H_v$-structure and which constitutes the generalization of algebraic hyperstructures like the hypergroups and the hyperrings. A particular case of this generalization is the very thin hyperstructure.
**Definition 5 (L. Konguetsof, Th. Vougiouklis, S. Spartalis [18]).** *A $H_v$-group is known as very thin if all its hyperoperations except one are singletons.*
With L. Konguetsof and S. Spartalis, Th. Vougiouklis [18] established the following proposition.
**Proposition 1 (L. Konguetsof, Th. Vougiouklis, S. Spartalis [18]).**

*Let be $(H, .)$ a finished very thin $H_v$-group of order $n > 1$. Let $a$ and $b$ be the only elements of $H$ such that $ab = A$ is of a strictly superior to 1 order.*

*(i) either for all $v$ de $H - \{a\}$; $va = a$ and two cases are to be considered: if $n = 2$, then there exists a group law $(*)$ on $H$, such that $a * b \in A$ and $x * y = xy$ for all $x$, $y$ of $H - \{(a, b)\}$,*
*if $n \geq 3$, then $a = b$, $H - \{a\}$ is a group, $A = H$ or $A = H - \{A\}$,*
*(ii) or there exists $v$ of $H$ such that $v \neq a$ and $va \neq a$, then there exists a group law $(*)$ almost associative on $H$ i.e. associativity is everywhere verified, except possibly for the triplets of elements where the product $a * b$ is such that $a * b \in A$, and $x * y = xy$ for all $x$, $y$ of $H - \{(a, b)\}$.*

So if we wish to characterize the very thin hypergroups it suffices to consider the first part of the proposition.

For $n = 3$, with the characterization L. Konguetsof - S. Spartalis - Th. Vougiouklis, we obtain two very thin hypergroups:

$$HF_1 = (a, b, c, b, ac, b, c, b, a)$$

and

$$HF_2 = (a, b, c, b, H, b, c, b, a)$$

In addition our results confirm, in an independent way, the result of Th. Vougiouklis [36], namely that there exists, at order 3, eight very thin $H_v$-groups with identity element. Moreover there exists, at order 3, 16 very thin $H_v$-groups.

## 4 Abelian $H_v$-groups of Order 4

In a previous note [7] we gave the number of Abelian hypergroups of order 4.
**Theorem 7 (R. Bayon-N. Lygeros [7, 4]).** *There exists, up to isomorphism, 10.614.362 abelian hypergroups of order 4.*
We then considered the case of the $H_v$-groups.

In the abelian case we have the following equivalence: $x(yz) \cap (xy)z \neq \emptyset \Leftrightarrow z(yx) \cap (zy)x \neq \emptyset$. What authorizes us to decrease the number of computations to check for weak associativity. As for the hypergroups, there exists at order 4 $15^{10}$ potential hyperoperations.
**Theorem 8 (R. Bayon-N. Lygeros [3, 1]).** *There exists, up to isomorphism, 8.028.299.905 abelian $H_v$-groups of order 4 (see table 5).*
We specify this result in the following table:

|  |  | Classes | | |
|---|---|---|---|---|
|  |  | Cyclics | non Cyclics | |
|  |  |  | Proj. | non Proj. |
| $|Aut(H_v)|$ | 1 | 5 | 3 | - |
|  | 2 | - | - | - |
|  | 3 | 38 | 5 | 6 |
|  | 4 | 582 | 22 | 39 |
|  | 6 | 2.215 | 45 | 144 |
|  | 8 | 2.149 | 39 | 144 |
|  | 12 | 1.859.161 | 1.827 | 39.773 |
|  | 24 | 7.994.020.227 | 86.159 | 32.287.322 |

Table 5

Classification of the abelian $H_v$-groups of order 4

**Theorem 9.** *There exists, up to isomorphism, 631.609 abelian $H_v$-groups of order 4 with scalar unit (see table 6).*

| $|Aut(H_v)|$ | 1 | 2 | 3 | 4 | 6 | 8 | 12 | 24 |
|---|---|---|---|---|---|---|---|---|
|  | - | - | - | 32 | - | 46 | 5.510 | 626.021 |

Table 6

Number of abelian $H_v$-groups of order 4 with scalar unit in respect with their automorphisms group

## 5   Hypergroups, $H_v$-groups and Posets

After enumerating hypergroups and $H_v$-groups, we construct the associated posets. We obtain the poset of hypergroups of order 2, the poset of hypergroups of order 3 and the poset of $H_v$-groups of order 2. R. Fraïssé and N. Lygeros like C. Chaunier and N. Lygeros have enumerated posets [9, 10, 11, 16]. R. Fraïssé and N. Lygeros also have studied representation by circle inclusion for small posets [16].

### 5.1   The Hypergroups

After having enumerated the hypergroups of order 2 (see table 1, annotation *),
we obtain the poset of hypergroups of order 2 and its circle order representation
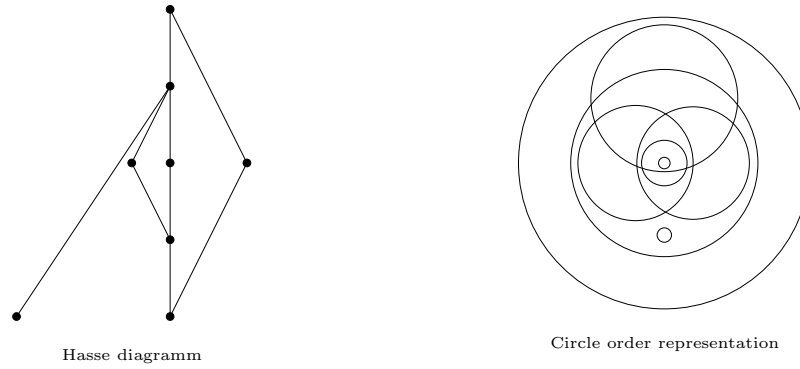
(see figure 1) [6].



Hasse diagramm                    Circle order representation

Fig. 1. Poset of hypergroups of order 2

In a previous article [6], we have studied the maximality of the longuest chain of posets of hypergroups of order $n$. A chain having the maximality property has a length $1 + (n - 1)n^2$. Thanks to an argument based on very thin hypergroups, we prove the following theorem:

**Theorem 10 (R. Bayon, N. Lygeros [6]).** *For $n \geq 3$, the maximal chain of the poset of the hypergroups has not the maximality property.*

We then construct the poset of hypergroups of order 3 (see table 7).

| Rank | # HG | Rank | # HG |
|------|------|------|------|
| 1 | 59 | 9 | 358 |
| 2 | 168 | 10 | 245 |
| 3 | 294 | 11 | 160 |
| 4 | 438 | 12 | 66 |
| 5 | 568 | 13 | 29 |
| 6 | 585 | 14 | 10 |
| 7 | 536 | 15 | 2 |
| 8 | 480 | 16 | 1 |

Table 7
Characteristics of the poset of hypergroups of order 3

*5.2 The $H_v$-groups*

The work on poset of hypergroups was extended by the construction of the poset of $H_v$-groups of order 2 (see figure 2).
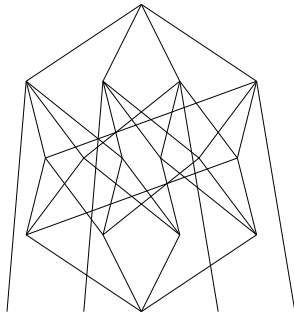
Fig. 2. Poset of $H_v$-groups of order 2

We construct the poset of very thin $H_v$-groups of order 3 too (see figure 3).
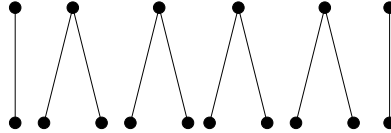


Fig. 3. Poset of the very thin $H_v$-groups of order 3

## 6 Extended Results

### 6.1 Hypocomplete Hypergroups

**Definition 6.** *An hyperproduct $xy$ of an hypergroup $(H, .)$ is complete if $xy = H$.*

**Definition 7.** *An hypergroup is hypocomplete when all its hyperoperation except one are complete.*

**Proposition 2.** *The hyperstructure defined by $aa = S \neq H$ et $\forall (x, y) \in H^2 \neq (a, a)$ $xy = H$ is an abelian hypocomplete hypergroup.*

*Proof.* • Obviously the hyperstructure is abelian and verifies reproductivity.
• The hyperstructure is associative :
$a(aa) = S = (aa)a$
$x(yz) = xH = H = Hz = (xy)z$ for $x, y, z \neq a$
$a(yz) = aH = H = Hz = (ay)z$ for $y, z \neq a$
$x(ya) = xH = H = Ha = (xy)a$ for $x, y \neq a$.

$\square$

**Theorem 11.** *The number of abelian hypocomplete hypergroup of order $n$ is $2(n-1)$ up to isomorphism.*

*Proof.* Let $(H, .)$ be an hypocomplete hypergroup of order $n$. Let assume that $aa$ is the noncomplete hyperproduct, and let $aa = S$ et $aa = R$ (generating hypergroups $H_S$ and $H_R$). We have $1 \leq S \leq n - 1$ et $1 \leq R \leq n - 1$. If $|S| \neq |R|$ then $H_R \not\cong H_S$. If $|S| = |R|$, there exist two equivalence classes :

- if $a \in S$ and $a \in R$ then $H_R \cong H_S$,
- if $a \in S$ and $a \notin R$ then $H_R \not\cong H_S$, that is isomorphic to $a \notin S$ and $a \in R$,
- if $a \notin S$ and $a \notin R$ then $H_R \cong H_S$.

Consequently there exist $2(n - 1)$ abelian hypocomplete hypergroups. $\qquad \square$

**Proposition 3.** *: The hyperstructure defined by $ab = S \neq H$ et $S \neq \{a\}$ ou $S \neq \{b\}$, et $\forall(x, y) \in H^2 \neq (a, b)$ $xy = H$ is an non-abelian hypocomplete hypergroup.*

*Proof.* • Obviously the hyperstructure is abelian and verifies reproductivity.
• The hyperstructure is associative :
$a(aa) = H = (aa)a$, $b(bb) = H = (bb)b$
$x(yz) = xH = H = Hz = (xy)z$ for $x, y, z \neq a, b$; for only one $a$ or $b$ in $x, y, z$ see previous proposition.
$a(bz) = aH = H = sz = (ab)z$ idem for the permutation; $a(yb) = aH = H = Hb = (ay)b$ idem for the permutation; $a(ba) = aH = H = Sa = (ab)a$;
$a(ab) = aS = H = Hb = (aa)b$

$\qquad \square$

**Remark 1.** *If $ab = a$ the hyperstructure is not associative : $(ab)b = ab = a \neq H = aH = a(bb)$. If $ab = b$ the hyperstructure is not associative : $(aa)b = Hb = H \neq b = ab = a(ab)$.*

**Theorem 12.** *The number of nonabelian hypocomplete hypergoup of order $n$ is $4(n - 2)$ up to isomorphism.*

*Proof.* Similar as theorem 11. $\qquad \square$

**Theorem 13.** *The number of hypocomplete hypergroup of order $n$ is $6n - 10$.* This theorem is the combination of theorem 11 and 12.

*6.2   Rigid Hypergroups and Rigid $H_v$-groups*

**Definition 8.** *Let be $(.)$ and $(*)$ two hyperoperations on $H$ we say $(*)$ is dual of $(.)$ if and only if $\forall x, y \in H x.y = y * x$. We note $(*) = d(.)$.*

**Proposition 4.** *$< H, . >$ is an hypergroup if and only if $< H, d(.) >$ is an hypergroup.*

**Proposition 5.** *$< H, . >$ is a $H_v$-group if and only if $< H, d(.) >$ is a $H_v$-group.*

**Proposition 6.** *If $< H, . >$ is a rigid quasigroup with $|H| > 2$, there exists only three possible squares :*

- $\forall x \in H,\ xx = x$
- $\forall x \in H,\ xx = H - \{x\}$
- $\forall x \in H,\ xx = H$

*Proof.* If there exists $x$ of $H$ such that $x \in xx$ then, by transposition, for all $x$ of $H$, $x$ belongs to $xx$.

In the same way, if there exists $x$ of $H$ such that $x \notin xx$ then for all $x$ of $H$, $x$ does not belong to $xx$.

If there exists $y \neq x$, such that $y$ belongs to $xx$. Let be $z$ different from $x$ and $y$; $xx = S \cup y$ and suppose that $z$ does not belong to $S$. Let be $f$ the transposition of $y$ and $z$, then $f$ induces a new labeling of $H$ ($xx = S \cup z$ because $f(S) = S$) : that contradicts the rigidity of $H$. Consequently if there exists $y$ different from $x$ with $y \in xx$, then all $y$ different from $x$ belongs to $xx$. $\square$

**Proposition 7.** *If $< H, . >$ is a rigid quasigroup with $|H| > 2$, there exists only seven possible cross products :*

(i) $\forall x, y \in H\ (x \neq y),\ xy = x$
(ii) $\forall x, y \in H\ (x \neq y),\ xy = y$
(iii) $\forall x, y \in H\ (x \neq y),\ xy = H - \{x\}$
(iv) $\forall x, y \in H\ (x \neq y),\ xy = H - \{y\}$
(v) $\forall x, y \in H\ (x \neq y),\ xy = H - \{x, y\}$
(vi) $\forall x, y \in H\ (x \neq y),\ xy = \{x, y\}$
(vii) $\forall x, y \in H\ (x \neq y),\ xy = H$

*Proof.* Suppose there exists $(x, y) \in H^2$ with $x \neq y$ and $x \in xy$. Let $z$ in $H$ and $f$ be the transposition of $x$ and $z$, then $z$ is in $zy$ because of definition 6. This time, let be $f$ the transposition of $z$ and $y$, then $x$ is in $xz$. So if there exists a $(x, y)$ in $H^2$ (with $x \neq y$) such that $x \in xy$ then for all $(x, y)$ in $H^2$ (with $x \neq y$), x belongs to $xy$. In the same way we show the following results :

- if there exists a $(x, y)$ in $H^2$ (with $x \neq y$) such that $y \in xy$ then for all $(x, y)$ in $H^2$ (with $x \neq y$), y belongs to $xy$.
- if there exists a $(x, y)$ in $H^2$ (with $x \neq y$) such that $x \notin xy$ then for all $(x, y)$ in $H^2$ (with $x \neq y$), x does not belong to $xy$.
- if there exists a $(x, y)$ in $H^2$ (with $x \neq y$) such that $y \notin xy$ then for all $(x, y)$ in $H^2$ (with $x \neq y$), y does not belong to $xy$.

Let $\alpha \in H$ with $\alpha \neq x$ and $\alpha \neq y$, so by rigidity $\alpha$ is in $xy$ (using the transposition of $\alpha$ and $z$). So $H - \{x, y\} \in xy$ and, by combination of proposition 1 and previous result, this implies that if there exists a $(x, y, z)$ in $H^3$ with $x \neq y$, $x \neq z$ and $y \neq z$ such that $z \in xy$ then $\forall x, y \; H - \{x, y\} \subset xy$.

The combination of the five previous results proves the current proposition. $\square$

We then lead an exhaustive study of the existence of the rigid hypergroups and $H_v$-groups. We summarize our results in table 8.

| x.x | x | y | $H - \{x\}$ | $H - \{y\}$ | $H - \{x, y\}$ | $\{x, y\}$ | $H$ |
|---|---|---|---|---|---|---|---|
| | | | | x.y | | | |
| $x$ | - | - | $H_{v1}$ | $d(H_{v1})$ | $Q_1$ | $H_1$ | $H_2$ |
| $H - \{x\}$ | - | - | - | - | - | $H_3$ | $H_4$ |
| $H$ | $H_{v2}$ | $d(H_{v2})$ | $H_{v3}$ | $d(H_{v3})$ | $H_{v4}$ | $H_5$ | $H_6$ |

Table 8
The 14 rigid Quasigroups.

**Proposition 8.** *$H_1$, $H_2$, $H_3$, $H_4$, $H_5$ and $H_6$ are hypergroups.*
**Proposition 9.** *$H_{vi}$, $d(H_{vi})$ (for $i \in [1, 4]$) are $H_v$-groups.*
**Proposition 10.** *$Q_1$ is a quasigroup at order 3 and a $H_v$-group at order greater than 3.*

## 7  Hyperrings

**Definition 9 (Th. Vougiouklis [32]).** *$(R, +, .)$ An hyperstructure is called an hyperring if $(R, +)$ is an hypergroup, $(R, .)$ is a semigroup and (.) is distributive in respect to (+).*
**Definition 10 (S. Spartalis, A.Dramalides and Th. Vougiouklis [28]).** *$(R, +, .)$ An hyperstructure is called an $H_v$-ring if $(R, +)$ is an $H_v$-group, $(R, .)$ is a weak semigroup and (.) is weakly distributive in respect to (+).*
A. Dramalidis enumerated a restricted class of $H_v$-rings, the dual $H_v$-rings.
**Definition 11 (A.Dramalidis [15]).** *An $H_v$-ring $(R, +, .)$ is dual if $(R, ., +)$ is an $H_v$-ring.*
He classified all $H_v$-ring such that $R = \{0, 1, a\}$ where 0 is the scalar unit of $H_v$-group $(R, +)$ and absorbing element of semi-hypergroup $(H, .)$ and 1 is the scalar unit of semi-hypergroup $(H, .)$. In the same way, he classified all hyperannoids, where (.) is not distributive in respect to (+). He needed to avoid a maximum of computations because they where done case by case. So he tried to minimize the role of associativity because of its high computational cost and use the symetry of duality.

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We enumerate hyperrings and $H_v$-rings of small orders and this will probably improve our understanding of the hyperannoids [8]. Indeed, certain categories of hypergroups were studied because of their low computational cost, but they were useless for the understanding of hyperstructures. Our research [6] already showed the greater importance of cyclic and single-power hypergroups than the canonical hypergroups [25, 26].

From an historical point of view, M. Krasner has introduced the notion of hyperring in 1966, ten years after the notion of hyperfield. So the hyperring in M. Krasner's sense generalizes his notion of hyperfield. This one was considered as the natural extension of F. Marty's hypergroups. But this extension is not as natural as it seems. In order to avoid technical problems, M. Krasner used *ad hoc* properties which were studied by its disciple J. Mittas. J. Mittas introduced canonical hypergroups which are, in short, a restriction of hyperring and consequently of hyperfield in Krasner's sense. This global schema seemed complete and closed, in fact no. The radically different approach of Th. Vougiouklis showed this critical point. Th. Vougiouklis started his work by weakening associativity in the hypergroup of F. Marty. It was then easy to extend this notion to hyperring and to hyperfield in a natural way. Moreover, this approach generalizes Krasner's and Rota's approaches. Th. Vougiouklis does not work in a specific case as canonical hypergroups. His approach is based on hypergroup in Marty's sense and moreover he introduces $H_v$-groups. He avoid the pitfall of representativity in the world of hypergroups. Indeed in our research, we show the low importance of canonical hypergroups in the set of hypergroups. From this observation, we easily deduce that M. Krasner's generalization on hyperring and hyperfield are analoguous in the corresponding world. So the generalization of Th. Vougiouklis embraces the whole set of hyperstructures. A natural approach to hyperrings is to construct them from their underlying hyperstructures. With this manner, we can easily check intermediate results. Consequently, we use the enumeration of hypergroups, semi-hypergroups, $H_v$-groups and $S_v$-groups (which are analogue of $H_v$-groups for semi-hypergroups).

**Proposition 11 (R. Bayon - N. Lygeros).** *Let $(R, +, .)$ be an hyperring then $Aut(R) = Aut(+) \cup Aut(.)$.*

**Corollary 1 (R. Bayon - N. Lygeros).** *Let $(R, +, .)$ be an hyperring then $|Aut(R)| \geq max(|Aut(+)|, |Aut(.)|)$.*

**Theorem 14 (R. Bayon - N. Lygeros).** *There are 63 isomorphism classes of hyperrings of order 2 (see table 4).*

|          |   | Classes |
|----------|---|---------|
| $|Aut(R)|$ | 1 | 6       |
|          | 2 | 114     |

Table 9
Classification of Hyperrings of Order 2

**Theorem 15 (R. Bayon - N. Lygeros).** *There are 875 isomorphism classes of $H_v$-rings of order 2 (see table 5).*

|          |   | Classes |
|----------|---|---------|
| $|Aut(R)|$ | 1 | 33 |
|          | 2 | 1684 |

Table 10
Classification of $H_v$-rings of Order 2

**Theorem 16 (R. Bayon - N. Lygeros).** *There are 33.277.642 isomorphism classes of hyperrings of order 3 (see table 6).*

|          |   | Classes |
|----------|---|---------|
| $|Aut(R)|$ | 1 | 31 |
|          | 2 | 506 |
|          | 3 | 67.857 |
|          | 6 | 199.528.434 |

Table 11
Classification of Hyperrings of Order 3

This global approach generalize the partial results obtained by Th. Vougiouklis and A. Dramalidis [15, 34].

## 8 Algorithm

### 8.1 Algorithm Structure

#### 8.1.1 Generation of Hyperstructures and Partitioning.

We generate the hyperstructures candidates by counting in base $n! - 1$. This counter enumerates all numbers with $n^2$ digits. During this generation, we prune candidates by verifying the axiom of reproduction. If the reproduction axiom is verified, we test the weak associativity. If the candidate has these two properties, it is a hyperstructure. We then determine its partition. We partition the hyperstructures in respect with the number of hyperproducts of a given order. The resulting partitioning is thin and uniform. With this partitioning we construct efficiently the poset of hyperstructures [6].

### 8.1.2 Isomorphism Test

**Definition 12.** *Two hyperstructures $< H, . >$ and $< H, * >$ are isomorphic if there exists $f \in Aut(H, *)$ such that $\forall (x, y) \in H^2\ xy = f(x * y)$.*
It is sufficient to pre-compute $S_n$ and to verify for each couple of $H_v$-groups $(< H, . >, < H, * >)$ if there exists $f \in S_n$ such that $f(< H, * >) = < H, . >$. In order to simplify the enumeration of hyperstructures, we only test isomorphism between hyperstructures of the same partition. We obtain the set of non isomorphic hyperstructures and the order of their automorphisms group.

### 8.1.3 Validation

With this algorithm we get the result of R. Migliorato [24], who computes the 23192 hypergroups of order 3, and the result of G. Nordo who [27] computes the 3999 non-isomorphic hypergroups of order 3.

### 8.1.4 An Enumeration Algorithm

Our previous algorithm is similar to G. Nordo's one, but our partitioning allows us to eliminate useless isomorphism tests. It is necessary in order to construct posets of hyperstructures, because we need to know all isomorphisms between hyperstructures. However we have developed a new algorithm for enumerative results. We generate all the hyperstructures, and for each of them we compute the order of its automorphisms group. The number of hyperstructures, up to isomorphism, $p$ is:

$$p = \sum_{i=1}^{n!} \frac{s_i}{i}$$

where $n$ is the order of the hyperstructures, and $s_i$ is the number of hyperstructures having an automorphisms group of order $i$. With this algorithm we get result at order 4.

### 8.1.5 Hyperrings

We generate all the simple hyperstructures (hypergroups, semigroups, $H_v$-groups,... ) and for each of them we compute the order of its automorphisms group. We check distributivity for each valid pair of hyperstructures. If the hyperringoid verifies distributivity, we compute and we store the order of their automorphisms group. As all paires have been checked, we determine the number of hyperrings, up to isomorphism.

# References

[1] R. Bayon and N. Lygeros. The $H_v$-groups and Marty-Moufang hypergroups. In *Proceedings of First International Conference on Algebraic Informatics*, pages 285–294. Aristotle University of Thessaloniki - Greece.

[2] R. Bayon and N. Lygeros. Les hypergroupes et $H_v$-groupes d'ordre 3. *Accepted in Bulletin of the Greek Mathematical Society.*

[3] R. Bayon and N. Lygeros. Number of abelian $H_v$-groups of order n. In N. J. A. Sloane, editor, *The On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/ njas/sequences/A108089, 2005.

[4] R. Bayon and N. Lygeros. Number of abelian hypergroups of order n. In N. J. A. Sloane, editor, *The On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/ njas/sequences/A132590, 2007.

[5] R. Bayon and N. Lygeros. Les hypergroupes d'ordre 3. *Italian Journal of Pure and Applied Mathematics*, 20:223–236, July 2006.

[6] R. Bayon and N. Lygeros. Catégories spécifiques d'hypergroupes d'ordre 3. In *Structure elements of hyper-structures, Xanthi*, pages 17–33. Spanidis, march 2005.

[7] R. Bayon and N. Lygeros. Les hypergroupes abéliens d'ordre 4. In *Structure elements of hyper-structures, Xanthi*, pages 35–39. Spanidis, march 2005.

[8] R. Bayon and N. Lygeros. Hyperstructures and automorphism groups. submitted.

[9] C. Chaunier and N. Lygeros. The number of orders with thirteen elements. *Order*, 9(3):203–204, 1992.

[10] C. Chaunier and N. Lygeros. Progrès dans l'énumération des posets. *C. R. Acad. Sci. Paris Sér. I Math.*, 314(10):691–694, 1992.

[11] C. Chaunier and N. Lygeros. Le nombre de posets à isomorphie près ayant 12 éléments. *Theoret. Comput. Sci.*, 123(1):89–94, 1994. Number theory, combinatorics and applications to computer science (Marseille, 1991).

[12] C. Chaunier and N. Lygeros. Posets minimaux ayant un groupe d'automorphismes d'ordre premier. *C. R. Acad. Sci. Paris Sér. I Math.*, 318(8):695–698, 1994.

[13] S-C. Chung and B-M. Choi. $H_v$-groups on the set $\{e, a, b\}$. *Italian Journal of Pure and Applied Mathematics*, 10:133–140, 2001.

[14] A. Dramaladis and Th. Vougiouklis. On a class of geometric fuzzy $H_v$-structures. In *8th Int. Congress on AHA Greece 2002*, pages 137–145, 2003.

[15] A. Dramalidis. Dual $H_v$-rings. *Rivista di mathematica pura ed applicata*, 17:55–62, 1995.

[16] R. Fraïssé and N. Lygeros. Petits posets: dénombrement, représentabilité par cercles et "compenseurs". *C. R. Acad. Sci. Paris Sér. I Math.*, 313(7):417–420, 1991.

[17] L. Konguetsof and Th. Vougiouklis. Constructions d'hyperanneaux. *Acta*

*Un. Car. Math. Ph.*, pages 9–13, 1987.

[18] L. Konguetsof, Th. Vougiouklis, and S. Spartalis. Sur les hyperstructures très fines. *Rendiconti di Matematica*, 13(VII):297–304, 1993.

[19] M. Kuntzmann. Opérations multiformes. hypergroupes. *C. R. Acad. Sci. Paris Math.*, 204:1787–1788, 1937.

[20] N. Lygeros and M. Mizony. Construction de posets dont le groupe d'automorphismes est isomorphe à un groupe donné. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(3):203–206, 1996.

[21] F. Marty. Sur une généralisation de la notion de groupe. In *8ème congrès des Mathématiciens Scandinaves, Stockholm*, pages 45–49, 1934.

[22] F. Marty. Rôle de la notion d'hypergroupe dans l'étude des groupes non abéliens. *C. R. Acad. Sci. Paris Math.*, 1935.

[23] F. Marty. Sur les groupes et hypergroupes attachés à une fraction rationnelle. *Annales scientifiques de l'E.N.S.*, 53:83–123, 1936.

[24] R. Migliorato. Ipergruppi di cardinalità 3 e isomorfismi di ipergruppoidi commutativi totalmente regolari. *Atti Convegno su Ipergruppi, altre Strutture Multivoche e loro applicazioni, Udine*, 1985.

[25] J. Mittas. Hypergroupes canoniques. *Mathematica Balkanica*, 2:165–179, 1972.

[26] J. Mittas. Hypergroupes renaissants. In *Atti Sem. Mat. Fis. Un. Modena*, volume 33, pages 319–328, 1984.

[27] G. Nordo. An algorithm on number of isomorphism classes of hypergroups of order 3. *Italian Journal of Pure and Applied Mathematics*, 1995.

[28] S. Spartalis, A.Dramalides, and Th. Vougiouklis. On $H_v$-group rings. *Algebras Groups and Geometries*, 15:47–54, 1988.

[29] S. Spartalis, A. Dramalidis, and Th. Vougiouklis. On $H_v$-group rings. *Algebras, Groups and Geometries*, 15:47–54, 1998.

[30] S. Spartalis and Th. Vougiouklis. The fundamental relations on $H_v$-rings. *Riv. Mat. Pura Appl.*, 14:7–20, 1994.

[31] Th. Vougiouklis. *Cyclicity of hypergroups*. PhD thesis, in greek, 1980.

[32] Th. Vougiouklis. The fundamental relation in hyperrings, the general hyperfield. In *Fourth INT. Congress on AHA*, pages 203–211, 1990.

[33] Th. Vougiouklis. The fundamental relation in hyperrings: The general hyperfield. pages 203–211, 1991.

[34] Th. Vougiouklis. *Hyperstructures and their Representations*. Hadronic Press, 1994.

[35] Th. Vougiouklis. A new class of hyperstructures. *Journal of combinatorics, information & system sciences*, 20:229–235, 1995.

[36] Th. Vougiouklis. $H_v$-groups defined on the same set. *Discrete Mathematics*, 155:259–265, 1996.

[37] Th. Vougiouklis. Fundamental relations in hyperstructures. *Bulletin of the greek mathematical society*, 42:113–118, 1999.

[38] Th. Vougiouklis. On hyperstructures obtained by attaching elements. In *Constantin Caratheodory in His... Origins Proceedings of the International Congress Vissa*, pages 197–206, 2001.

[39] Th. Vougiouklis. The $h/v$-structures. *Journal of Discrete Mathematical Sciences & Cryptography*, 6:235–243, 2003.

[40] Th. Vougiouklis, S. Spartalis, and M. Kessoglides. Weak hyperstructures on small sets. *Ratio Mathematica*, 12:90–96, 1997.

[41] H.S. Wall. Hypergroups. *American Journal of Mathematics*, 59:78–98, 1937.